FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

Version 0
September 17, 2008

STOP!!!!

Did you notice the FREEDOMS we've granted you?

Our copyright license is **different!** It grants freedoms rather than just imposing restrictions like all those other textbooks you probably own or use.

- We know people copy textbooks illegally but we would LOVE it if you copied our's - go ahead copy to your hearts content, **legally!**
- Publishers revenue is generated by controlling the market, we don't want any money, go ahead, distribute our books far and wide - we DARE you!
- Ever wanted to change your textbook? Of course you have! Go ahead change ours, make your own version, get your friends together, rip it apart and put it back together the way you like it. That's what we really want!
- Copy, modify, adapt, enhance, share, critique, adore, and contextualise. Do it all, do it with your colleagues, your friends or alone but get involved! Together we can overcome the challenges our complex and diverse country presents.
- So what is the catch? The only thing you can't do is take this book, make a few changes and then tell others that they can't do the same with your changes. It's share and share-alike and we know you'll agree that is only fair.
- These books were written by volunteers who want to help support education, who want the facts to be freely available for teachers to copy, adapt and re-use. Thousands of hours went into making them and they are a gift to everyone in the education community.

FHSST Core Team

Mark Horner ; Samuel Halliday ; Sarah Blyth ; Rory Adams ; Spencer Wheaton

FHSST Editors

Jaynie Padayachee ; Joanne Boulle ; Diana Mulcahy ; Annette Nell ; René Toerien ; Donovan Whitfield

FHSST Contributors

Rory Adams ; Prashant Arora ; Richard Baxter ; Dr. Sarah Blyth ; Sebastian Bodenstein ; Graeme Broster ; Richard Case ; Brett Cocks ; Tim Crombie ; Dr. Anne Dabrowski ; Laura Daniels ; Sean Dobbs ; Fernando Durrell ; Dr. Dan Dwyer ; Frans van Eeden ; Giovanni Franzoni ; Ingrid von Glehn ; Tamara von Glehn ; Lindsay Glesener ; Dr. Vanessa Godfrey ; Dr. Johan Gonzalez ; Hemant Gopal ; Umeshree Govender ; Heather Gray ; Lynn Greeff ; Dr. Tom Gutierrez ; Brooke Haag ; Kate Hadley ; Dr. Sam Halliday ; Asheena Hanuman ; Neil Hart ; Nicholas Hatcher ; Dr. Mark Horner ; Mfandaidza Hove ; Robert Hovden ; Jennifer Hsieh ; Clare Johnson ; Luke Jordan ; Tana Joseph ; Dr. Jennifer Klay ; Lara Kruger ; Sihle Kubheka ; Andrew Kubik ; Dr. Marco van Leeuwen ; Dr. Anton Machacek ; Dr. Komal Maheshwari ; Kosma von Maltitz ; Nicole Masureik ; John Mathew ; JoEllen McBride ; Nikolai Meures ; Riana Meyer ; Jenny Miller ; Abdul Mirza ; Asogan Moodaly ; Jothi Moodley ; Nolene Naidu ; Tyrone Negus ; Thomas O'Donnell ; Dr. Markus Oldenburg ; Dr. Jaynie Padayachee ; Nicolette Pekeur ; Sirika Pillay ; Jacques Plaut ; Andrea Prinsloo ; Joseph Raimondo ; Sanya Rajani ; Prof. Sergey Rakityansky ; Alastair Ramlakan ; Razvan Remsing ; Max Richter ; Sean Riddle ; Evan Robinson ; Dr. Andrew Rose ; Bianca Ruddy ; Katie Russell ; Duncan Scott ; Helen Seals ; Ian Sherratt ; Roger Sieloff ; Bradley Smith ; Greg Solomon ; Mike Stringer ; Shen Tian ; Robert Torregrosa ; Jimmy Tseng ; Helen Waugh ; Dr. Dawn Webber ; Michelle Wen ; Dr. Alexander Wetzler ; Dr. Spencer Wheaton ; Vivian White ; Dr. Gerald Wigger ; Harry Wiggins ; Wendy Williams ; Julie Wilson ; Andrew Wood ; Emma Wormauld ; Sahal Yacoob ; Jean Youssef

Contributors and editors have made a sincere effort to produce an accurate and useful resource. Should you have suggestions, find mistakes or be prepared to donate material for inclusion, please don't hesitate to contact us. We intend to work with all who are willing to help make this a continuously evolving resource!

www.fhsst.org

Contents

Chapter 13

Geometry - Grade 10

13.1 Introduction

Geometry (Greek: geo = earth, metria = measure) arose as the field of knowledge dealing with spatial relationships. It was one of the two fields of pre-modern mathematics, the other being the study of numbers. In modern times, geometric concepts have become very complex and abstract and are barely recognizable as the descendants of early geometry.

Activity :: Researchproject : History of Geometry

Work in pairs or groups and investigate the history of the foundation of geometry. Describe the various stages of development and how the following cultures used geometry to improve their lives.

1. Ancient Indian geometry (c. 3000 - 500 B.C.)
   (a) Harappan geometry
   (b) Vedic geometry
2. Classical Greek geometry (c. 600 - 300 B.C.)
   (a) Thales and Pythagoras
   (b) Plato
3. Hellenistic geometry (c. 300 B.C - 500 C.E.)
   (a) Euclid
   (b) Archimedes

13.2 Right Prisms and Cylinders

In this section we study how to calculate the surface areas and volumes of right prisms and cylinders. A right prism is a polygon that has been stretched out into a tube so that the height of the tube is perpendicular to the base. A square prism has a base that is a square and a triangular prism has a base that is a triangle.

It is relatively simple to calculate the surface areas and volumes of prisms.

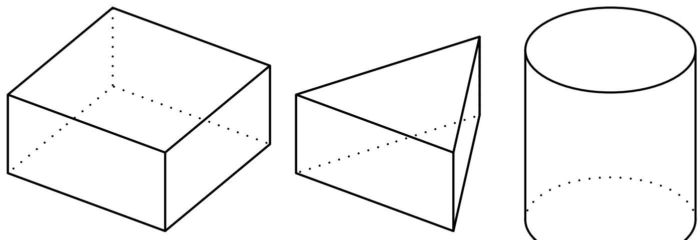


Figure 13.1: Examples of a right square prism, a right triangular prism and a cylinder.

13.2.1 Surface Area

The term *surface area* refers to the total area of the exposed or outside surfaces of a prism. This is easier to understand if you imagine the prism as a solid object.

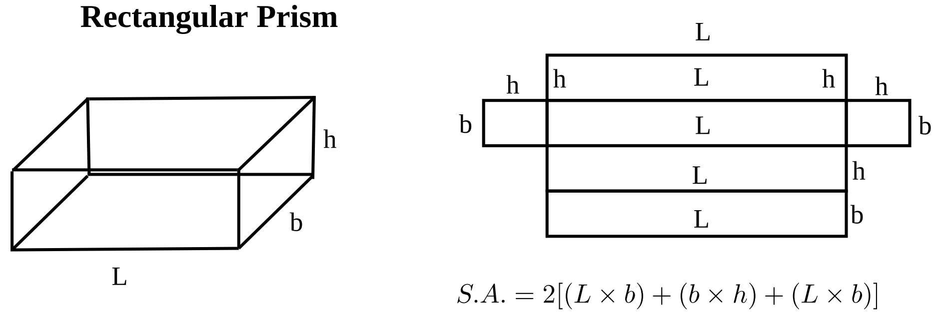
If you examine the prisms in Figure 13.1, you will see that each face of a prism is a simple polygon. For example, the triangular prism has two faces that are triangles and three faces that are rectangles. Therefore, in order to calculate the surface area of a prism you simply have to calculate the area of each face and add it up. In the case of a cylinder the top and bottom faces are circles, while the curved surface flattens into a rectangle.

Surface Area of Prisms
Calculate the area of each face and add the areas together to get the surface area.

Activity :: Discussion : surface areas

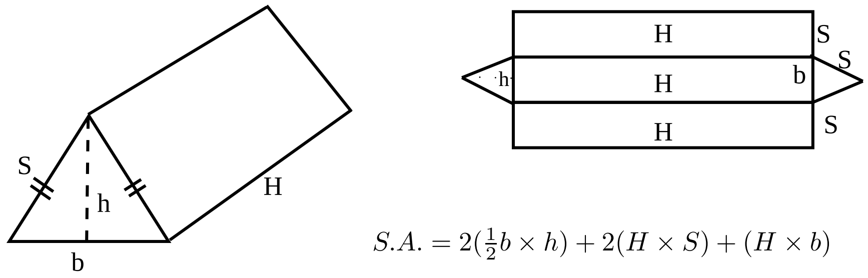
Study the following prisms, nets and formulae. Explain to your partner, how each relates to the other.

Rectangular Prism



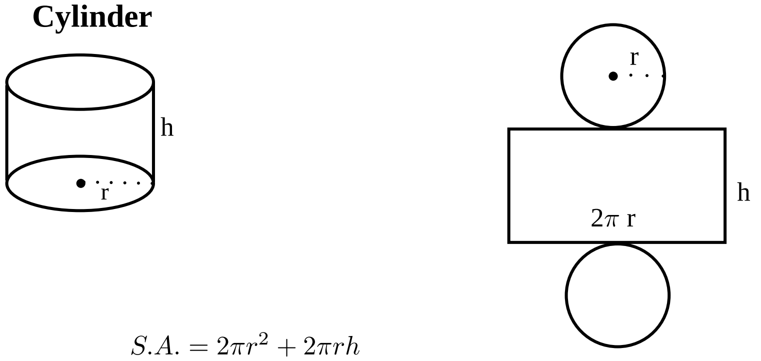
$S.A. = 2[(L \times b) + (b \times h) + (L \times b)]$

Triangular Prism



$S.A. = 2(\frac{1}{2}b \times h) + 2(H \times S) + (H \times b)$

Cylinder



$S.A. = 2\pi r^2 + 2\pi rh$

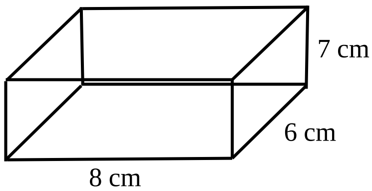
?

Exercise: Surface areas

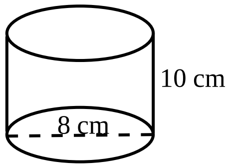
1. Calculate the surface area in each of the following:

•

1.

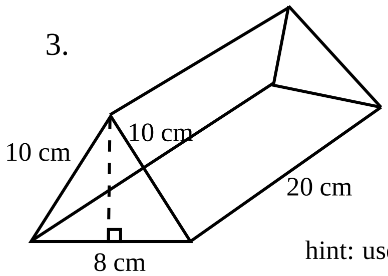


2.



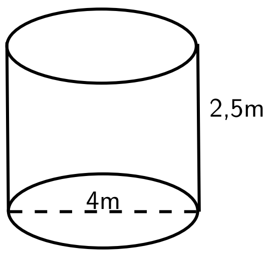
hint: diameter = 2 × radius

3.



hint: use Pythagoras to find height of triangular face.

2. If a litre of paint, paints $2m^2$, how much paint is needed to paint:
   (a) A rectangular swimming pool with dimensions $4m \times 3m \times 2{,}5m$, inside walls and floor only.
   (b) The inside walls and floor of a circular reservoir with diameter $4m$ and height $2{,}5m$



13.2.2 Volume

The volume of a right prism is calculated by multiplying the area of the base by the height. So, for a square prism of side length a and height h the volume is $a \times a \times h = a^2h$.

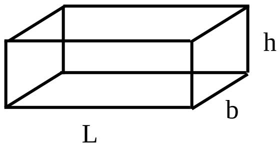
Volume of Prisms
Calculate the area of the base and multiply by the height to get the volume of a prism.

?

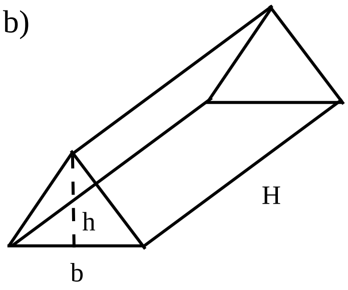
Exercise: Volume

1. Write down the formula for each of the following volumes:

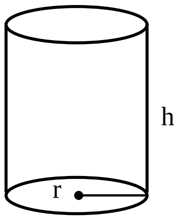
a)



b)

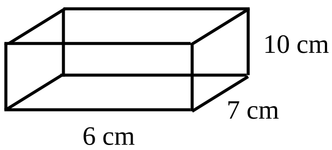


c)

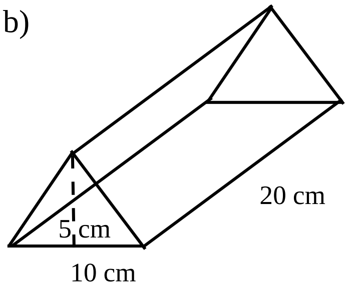


2. Calculate the following volumes:

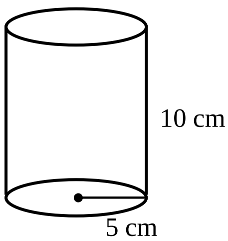
a)



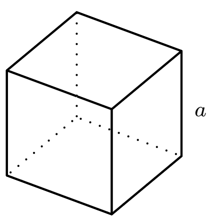
b)



c)

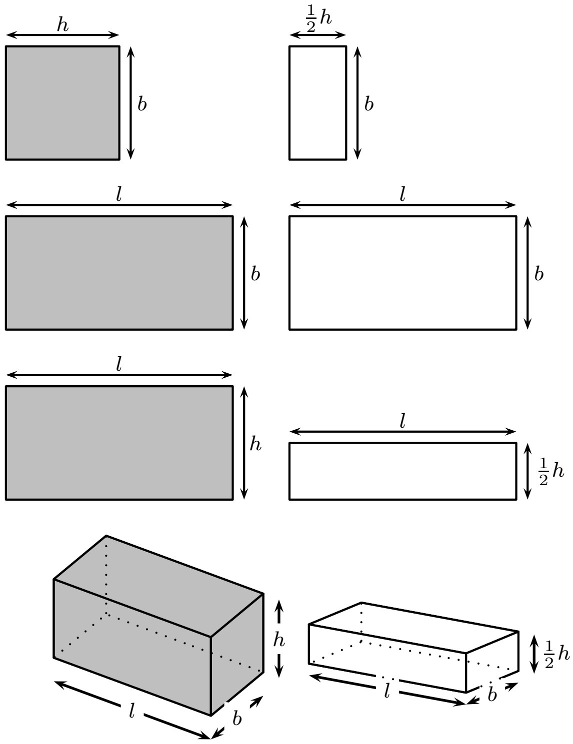


3. A cube is a special prism that has all edges equal. This means that each face is a square. An example of a cube is a die. Show that for a cube with side length a, the surface area is $6a^2$ and the volume is a^3.



Now, what happens to the surface area if one dimension is multiplied by a constant? For example, how does the surface area change when the height of a rectangular prism is divided by 2?

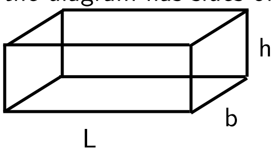
Worked Example 50: Scaling the dimensions of a prism



Surface Area $= 2(l \times h + l \times b + b \times h)$	Surface Area $= 2(l \times \frac{1}{2}h + l \times b + b \times \frac{1}{2}h)$
Volume $= l \times b \times h$	Volume $= l \times b \times \frac{1}{2}h$ $= \frac{1}{2}(l \times b \times h)$

Figure 13.2: Rectangular prisms

Question: The size of a prism is specified by the length of its sides. The prism in the diagram has sides of lengths L, b and h.



a) Consider enlarging all sides of the prism by a constant factor x. Where $x > 1$. Calculate the volume and surface area of the enlarged prism as a function of the factor x and the volume of the original volume.

a) In the same way as above now consider the case, where $0 < x < 1$. Now calculate the reduction factor in the volume and the surface area.

Answer

Step 1 : Identify

The volume of a prism is given by:

$V = L \times b \times h$

The surface area of the prism is given by:

$A = 2 \times (L \times b + L \times h + b \times h)$

Step 2 : Rescale

If all the sides of the prism get rescaled, the new sides will be:

$$\begin{aligned} L' &= x \times L \\ b' &= x \times b \\ h' &= x \times h \end{aligned}$$

The new volume will then be given by:

$$\begin{aligned} V' &= L' \times b' \times h' \\ &= x \times L \times x \times b \times x \times h \\ &= x^3 \times L \times b \times h \\ &= x^3 \times V \end{aligned}$$

The new surface area of the prism will be given by:

$$\begin{aligned} A' &= 2 \times (L' \times b' + L' \times h' + b' \times h') \\ &= 2 \times (x \times L \times x \times b + x \times L \times x \times h + x \times b \times x \times h) \\ &= x^2 \times 2 \times (L \times b + L \times h + b \times h) \\ &= x^2 \times A \end{aligned}$$

Step 3 : Interpret

a) We found above that the new volume is given by:
$V' = x^3 \times V$
Since $x > 1$, the volume of the prism will be increased by a factor of x^3.
The surface area of the rescaled prism was given by:
$A' = x^2 \times A$
Again, since $x > 1$, the surface area will be increased by a factor of x^2.

b) The answer here is based on the same ideas as above.
In analogy, since here $0 < x < 1$, the volume will de reduced by a factoe of x^3 and the surface area will be decreased by a factor of x^2

When the length of one of the sides is multiplied by a constant the effect is to multiply the original volume by that constant, as for the example in Figure 13.2.

13.3 Polygons

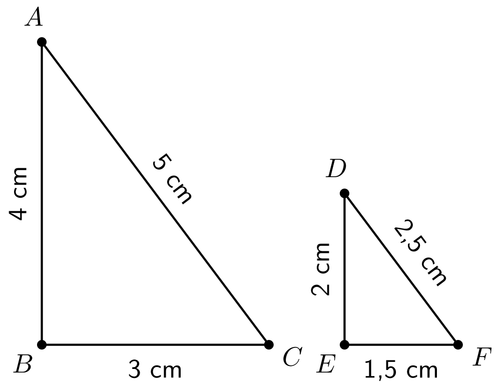
Polygons are all around us. A stop sign is in the shape of an octagon, an eight-sided polygon. The honeycomb of a beehive consists of hexagonal cells.

In this section, you will learn about similar polygons.

13.3.1 Similarity of Polygons

Activity :: Discussion : Similar Triangles
Fill in the table using the diagram and then answer the questions that follow.

$\frac{\mathrm{AB}}{\mathrm{DE}} = \frac{...cm}{...cm} = ...$	$\hat{A} = ...^\circ$	$\hat{D} ...^\circ$
$\frac{\mathrm{BC}}{\mathrm{EF}} = \frac{...cm}{...cm} = ...$	$\hat{B} = ...^\circ$	$\hat{E} = ...^\circ$
$\frac{\mathrm{AC}}{\mathrm{DF}} = \frac{...cm}{...cm} = ...$	$\hat{C} ...^\circ$	$\hat{F} = ...^\circ$



1. What can you say about the numbers you calculated for: $\frac{\mathrm{AB}}{\mathrm{DE}}$, $\frac{\mathrm{BC}}{\mathrm{EF}}$, $\frac{\mathrm{AC}}{\mathrm{DF}}$?
2. What can you say about $\hat{A}$ and $\hat{D}$?
3. What can you say about $\hat{B}$ and $\hat{E}$?
4. What can you say about $\hat{C}$ and $\hat{F}$?

If two polygons are *similar*, one is an enlargement of the other. This means that the two polygons will have the same angles and their sides will be in the same proportion.

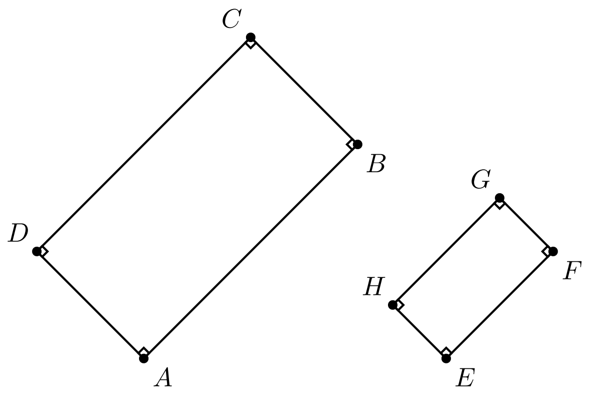
We use the symbol $\equiv$ to mean *is similar to*.

Definition: Similar Polygons
Two polygons are similar if:

1. their corresponding angles are equal, or
2. the ratios of corresponding sides are equal.

Worked Example 51: Similarity of Polygons

Question: Show that the following two polygons are similar.



Answer
Step 1 : Determine what is required
We are required to show that the pair of polygons is similar. We can do this by showing that the ratio of corresponding sides is equal or by showing that corresponding angles are equal.
Step 2 : Decide how to approach the problem

We are not given the lengths of the sides, but we are given the angles. So, we can show that corresponding angles are equal.

Step 3 : Show that corresponding angles are equal

All angles are given to be 90° and

$$\begin{aligned}\hat{A} &= \hat{E} \\ \hat{B} &= \hat{F} \\ \hat{C} &= \hat{G} \\ \hat{D} &= \hat{H}\end{aligned}$$

Step 4 : Final answer

Since corresponding angles are equal, the polygons ABCD and EFGH are similar.

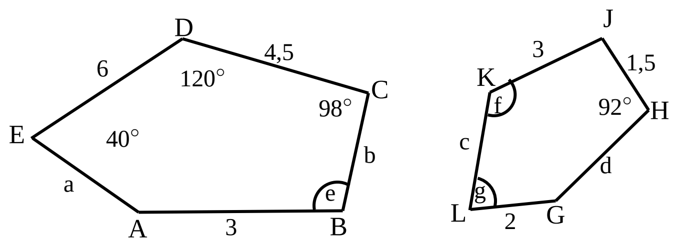
Step 5 : Comment on result

This result shows that all rectangles are similar to each other, because all rectangles will always have corresponding angles equal to each other.

Important: All rectangles and squares are similar.

Worked Example 52: Similarity of Polygons

Question: If two pentagons ABCDE and GHJKL are similar, determine the lengths of the sides and angles labelled with letters:



Answer

Step 1 : Determine what is given

We are given that ABCDE and GHJKL are similar. This means that:

$$\frac{\text{AB}}{\text{GH}} = \frac{\text{BC}}{\text{HJ}} = \frac{\text{CD}}{\text{JK}} = \frac{\text{DE}}{\text{KL}} = \frac{\text{EA}}{\text{LG}}$$

and

$$\begin{aligned}\hat{A} &= \hat{G} \\ \hat{B} &= \hat{H} \\ \hat{C} &= \hat{J} \\ \hat{D} &= \hat{K} \\ \hat{E} &= \hat{L}\end{aligned}$$

Step 2 : Determine what is required

We are required to determine the following lengths:

1. a, b, c and d

and the following angles:

1. e, f and g

Step 3 : Decide how to approach the problem

The corresponding angles are equal, so no calculation is needed. We are given one pair of sides DC and KJ that correspond $\frac{DC}{KJ} = \frac{4,5}{3} = 1,5$ so we know that all sides of $KJHGL$ are 1,5 times smaller than $ABCDE$.

Step 4 : Calculate lengths

$$\begin{aligned} \frac{a}{2} &= 1,5 \quad \therefore \quad a = 2 \times 1,5 = 3 \\ \frac{b}{1,5} &= 1,5 \quad \therefore \quad b = 1,5 \times 1,5 = 2,25 \\ \frac{6}{c} &= 1,5 \quad \therefore \quad c = 6 \div 1,5 = 4 \\ \frac{3}{d} &= 1,5 \quad \therefore \quad d = 2 \end{aligned}$$

Step 5 : Calculate angles

$$\begin{aligned} e &= 92^\circ \text{(corresponds to H)} \\ f &= 120^\circ \text{(corresponds to D)} \\ g &= 40^\circ \text{(corresponds to E)} \end{aligned}$$

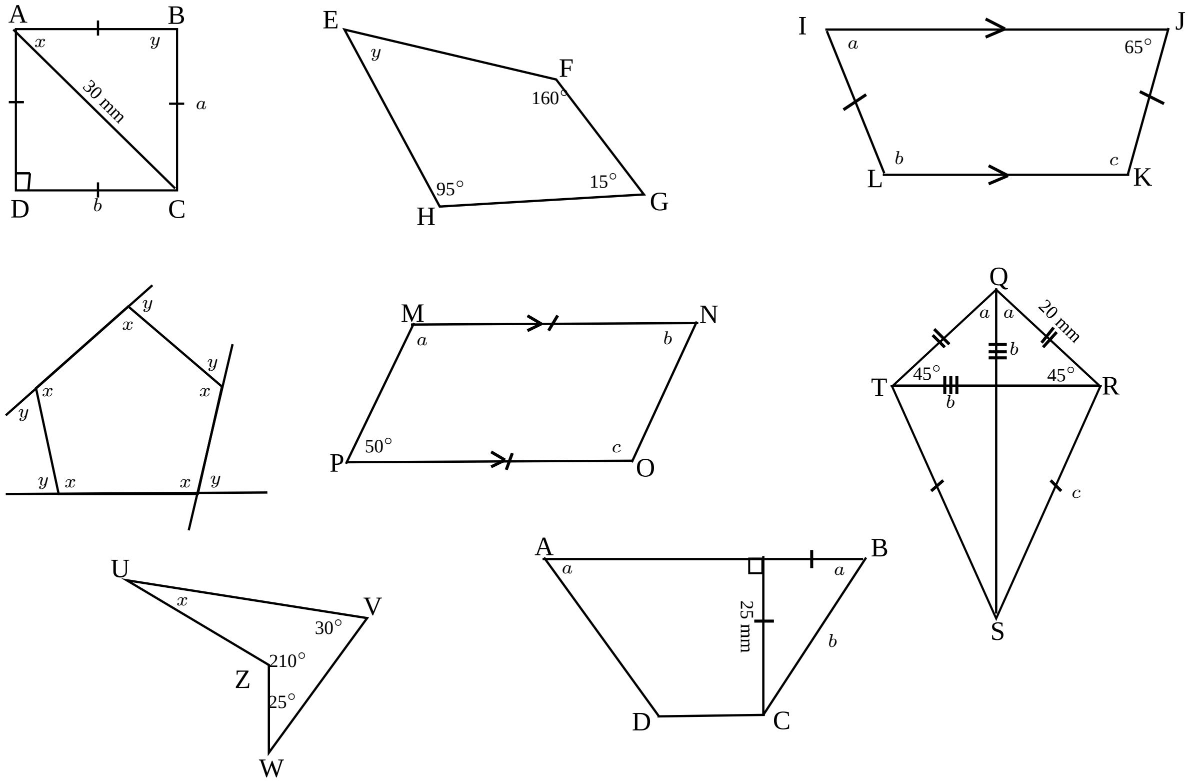
Step 6 : Write the final answer

$$\begin{aligned} a &= 3 \\ b &= 2,25 \\ c &= 4 \\ d &= 2 \\ e &= 92^\circ \\ f &= 120^\circ \\ g &= 40^\circ \end{aligned}$$

Activity :: Similarity of Equilateral Triangles : Working in pairs, show that all equilateral triangles are similar.

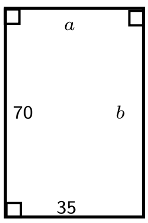
Exercise: Polygons-mixed

1. Find the values of the unknowns in each case. Give reasons.

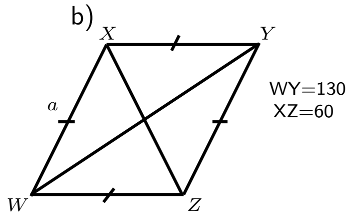


2. Find the angles and lengths marked with letters in the following figures:

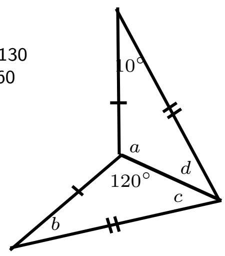
a)



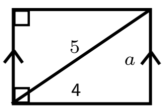
b)



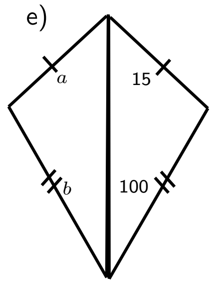
c)



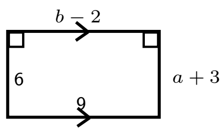
d)



e)



f)



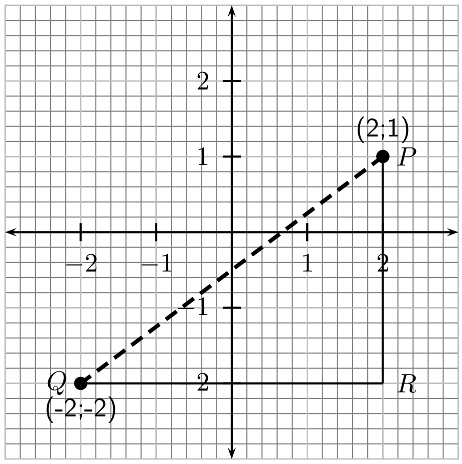
13.4 Co-ordinate Geometry

13.4.1 Introduction

Analytical geometry, also called co-ordinate geometry and earlier referred to as Cartesian geometry, is the study of geometry using the principles of algebra, and the Cartesian co-ordinate system. It is concerned with defining geometrical shapes in a numerical way, and extracting numerical information from that representation. Some consider that the introduction of analytic geometry was the beginning of modern mathematics.

13.4.2 Distance between Two Points

One of the simplest things that can be done with analytical geometry is to calculate the distance between two points. *Distance* is a number that describes how far apart two point are. For example, point P has co-ordinates $(2;1)$ and point Q has co-ordinates $(-2;-2)$. How far apart are points A and B? In the figure, this means how long is the dashed line?

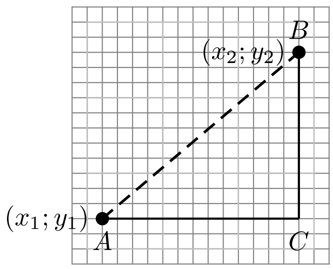


In the figure, it can be seen that the length of the line PR is 3 units and the length of the line QR is four units. However, the $\triangle PQR$, has a right angle at R. Therefore, the length of the side PQ can be obtained by using the Theorem of Pythagoras:

$$\begin{aligned} PQ^2 &= PR^2 + QR^2 \\ \therefore PQ^2 &= 3^2 + 4^2 \\ \therefore PQ &= \sqrt{3^2 + 4^2} = 5 \end{aligned}$$

The length of AB is the distance between the points A and B.

In order to generalise the idea, assume A is any point with co-ordinates $(x_1;y_1)$ and B is any other point with co-ordinates $(x_2;y_2)$.



The formula for calculating the distance between two points is derived as follows. The distance between the points A and B is the length of the line AB. According to the Theorem of Pythagoras, the length of AB is given by:

$$AB = \sqrt{AC^2 + BC^2}$$

However,

$$\begin{aligned} BC &= y_2 - y_1 \\ AC &= x_2 - x_1 \end{aligned}$$

Therefore,

$$\begin{aligned} AB &= \sqrt{AC^2 + BC^2} \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \end{aligned}$$

Therefore, for any two points, $(x_1;y_1)$ and $(x_2;y_2)$, the formula is:

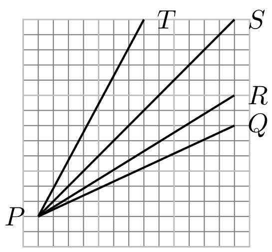
$$\boxed{\mathrm{Distance} = \sqrt{(\mathrm{x_1}-\mathrm{x_2})^2+(\mathrm{y_1}-\mathrm{y_2})^2}}$$

Using the formula, distance between the points P and Q with co-ordinates (2;1) and (-2;-2) is then found as follows. Let the co-ordinates of point P be $(x_1;y_1)$ and the co-ordinates of point Q be $(x_2;y_2)$. Then the distance is:

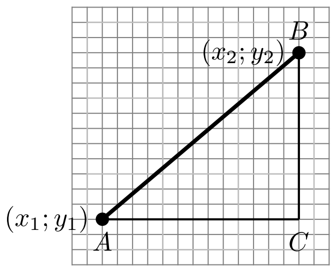
$$\begin{aligned}\text{Distance} &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\ &= \sqrt{(2-(-2))^2+(1-(-2))^2}\\ &= \sqrt{(2+2)^2+(1+2)^2}\\ &= \sqrt{16+9}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

13.4.3 Calculation of the Gradient of a Line

The gradient of a line describes how steep the line is. In the figure, line PT is the steepest. Line PS is less steep than PT but is steeper than PR, and line PR is steeper than PQ.



The gradient of a line is defined as the ratio of the vertical distance to the horizontal distance. This can be understood by looking at the line as the hypotenuse of a right-angled triangle. Then the gradient is the ratio of the length of the vertical side of the triangle to the horizontal side of the triangle. Consider a line between a point A with co-ordinates $(x_1;y_1)$ and a point B with co-ordinates $(x_2;y_2)$.



$$\boxed{\mathrm{Gradient} = \frac{\mathrm{y_2}-\mathrm{y_1}}{\mathrm{x_2}-\mathrm{x_1}}}$$

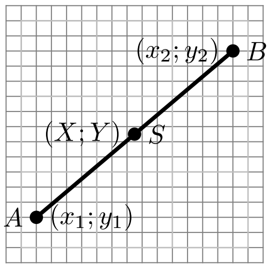
For example the gradient of the line between the points P and Q, with co-ordinates (2;1) and (-2;-2) (Figure 13.4.2) is:

$$\begin{aligned}\text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-2-1}{-2-2}\\ &= \frac{-3}{-4}\\ &= \frac{3}{4}\end{aligned}$$

13.4.4 Midpoint of a Line

Sometimes, knowing the co-ordinates of the middle point or *midpoint* of a line is useful. For example, what is the midpoint of the line between point P with co-ordinates $(2;1)$ and point Q with co-ordinates $(-2;-2)$.

The co-ordinates of the midpoint of any line between any two points A and B with co-ordinates $(x_1;y_1)$ and $(x_2;y_2)$, is generally calculated as follows. Let the midpoint of AB be at point S with co-ordinates $(X;Y)$. The aim is to calculate X and Y in terms of $(x_1;y_1)$ and $(x_2;y_2)$.



$$\begin{aligned}X &= \frac{x_1+x_2}{2}\\ Y &= \frac{y_1+y_2}{2}\\ \therefore \quad & S\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)\end{aligned}$$

Then the co-ordinates of the midpoint (S) of the line between point P with co-ordinates $(2;1)$ and point Q with co-ordinates $(-2;-2)$ is:

$$\begin{aligned}X &= \frac{x_1+x_2}{2}\\ &= \frac{-2+2}{2}\\ &= 0\\ Y &= \frac{y_1+y_2}{2}\\ &= \frac{-2+1}{2}\\ &= -\frac{1}{2}\end{aligned}$$

$$\therefore S(0;-\frac{1}{2})$$

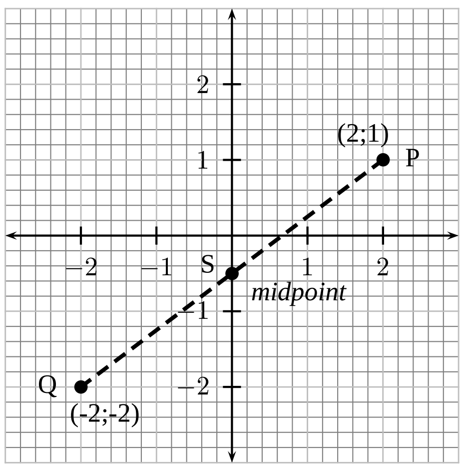
It can be confirmed that the distance from the each end point to the midpoint is equal. The co-ordinate of the midpoint S is $(0;-0{,}5)$.

$$\begin{aligned}
PS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(0 - 2)^2 + (-0.5 - 1)^2} \\
&= \sqrt{(-2)^2 + (-1.5)^2} \\
&= \sqrt{4 + 2.25} \\
&= \sqrt{6.25}
\end{aligned}$$

and

$$\begin{aligned}
QS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{(0 - (-2))^2 + (-0.5 - (-2))^2} \\
&= \sqrt{(0 + 2))^2 + (-0.5 + 2))^2} \\
&= \sqrt{(2))^2 + (-1.5))^2} \\
&= \sqrt{4 + 2.25} \\
&= \sqrt{6.25}
\end{aligned}$$

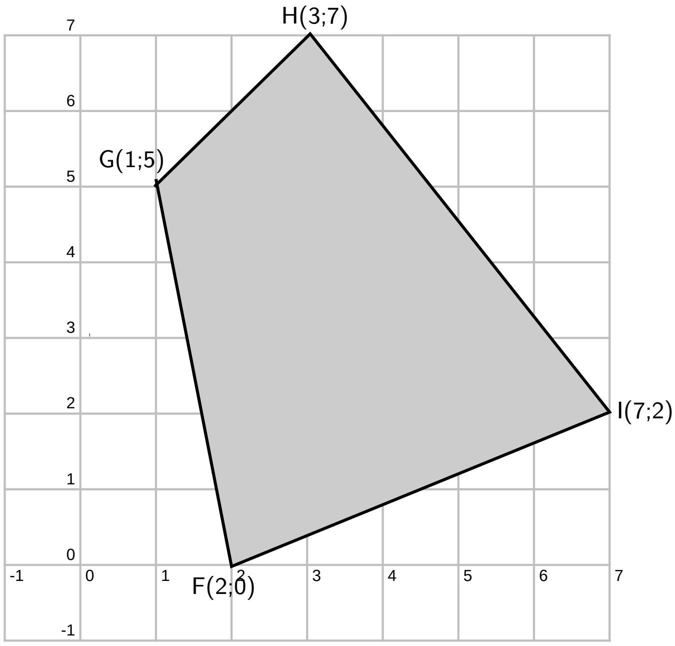
It can be seen that $PS = QS$ as expected.



?

Exercise: Co-ordinate Geometry

1. In the diagram given the vertices of a quadrilateral are F(2;0), G(1;5), H(3;7)



and I(7;2).

a) What are the lengths of the opposite sides of FGHI?
b) Are the opposite sides of FGHI parallel?
c) Do the diagonals of FGHI bisect each other?
d) Can you state what type of quadrilateral FGHI is? Give reasons for your answer.

2. A quadrialteral ABCD with vertices A(3;2), B(1;7), C(4;5) and D(1;3) is given.

a) Draw the qaudrilateral.
b) Find the lengths of the sides of the quadrilateral.

3. S(1;4), T(-1;2), U(0;-1) and V(4;-1) are the vertices of a pentagon.

a) Are two of the sides of this pentagon parallel? If yes, find them.
b) Are two of the sides of this pentagon of equal length? If yes, find them.

4. ABCD is a quadrilateral with verticies A(0;3), B(4;3), C(5;-1) and D(-1;-1).

a) Show that:
(i) $AD = BC$
(ii) $AB \parallel DC$
b) What name would you give to ABCD?
c) Show that the diagonals AC and BD do not bisect each other.

5. P, Q, R and S are the points (-2;0), (2;3), (5;3), (-3;-3) respectively.

a) Show that:
(i) $SR = 2PQ$
(ii) $SR \parallel PQ$
b) Calculate:
(i) PS
(ii) QR
c) What kind of a quadrilateral is PQRS? Give reasons for your answers.

6. EFGH is a parallelogram with verticies E(-1;2), F(-2;-1) and G(2;0). Find the co-ordinates of H by using the fact that the diagonals of a parallelogram bisect each other.

13.5 Transformations

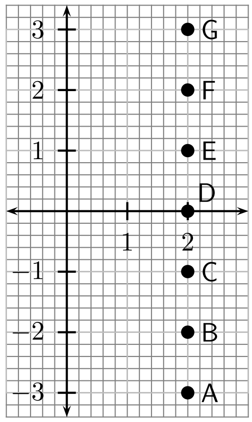
In this section you will learn about how the co-ordinates of a point change when the point is moved horizontally and vertically on the Cartesian plane. You will also learn about what happens to the co-ordinates of a point when it is reflected on the x-axis, y-axis and the line $y = x$.

13.5.1 Translation of a Point

When something is moved in a straight line, we say that it is *translated*. What happens to the co-ordinates of a point that is translated horizontally or vertically?

Activity :: Discussion : Translation of a Point Vertically

Complete the table, by filling in the co-ordinates of the points shown in the figure.



Point	x co-ordinate	y co-ordinate
A		
B		
C		
D		
E		
F		
G		

What do you notice about the x co-ordinates? What do you notice about the y co-ordinates?
What would happen to the co-ordinates of point A, if it was moved to the position of point G?

When a point is moved vertically up or down on the Cartesian plane, the x co-ordinate of the point remains the same, but the y co-ordinate changes by the amount that the point was moved up or down.

For example, in Figure 13.3 Point A is moved 4 units upwards to the position marked by G. The new x co-ordinate of point A is the same (x=1), but the new y co-ordinate is shifted in the positive y direction 4 units and becomes y=-2**+4**=2. The new co-ordinates of point A are therefore G(1;2). Similarly, for point B that is moved downwards by 5 units, the x co-ordinate is the same ($x = -2{,}5$), but the y co-ordinate is shifted in the negative y-direction by 5 units. The new y co-ordinate is therefore y=2,5 **-5**=-2,5.

! **Important:** If a point is shifted upwards, the new y co-ordinate is given by adding the shift to the old y co-ordinate. If a point is shifted downwards, the new y co-ordinate is given by subtracting the shift from the old y co-ordinate.

Activity :: Discussion : Translation of a Point Horizontally

Complete the table, by filling in the co-ordinates of the points shown in the figure.

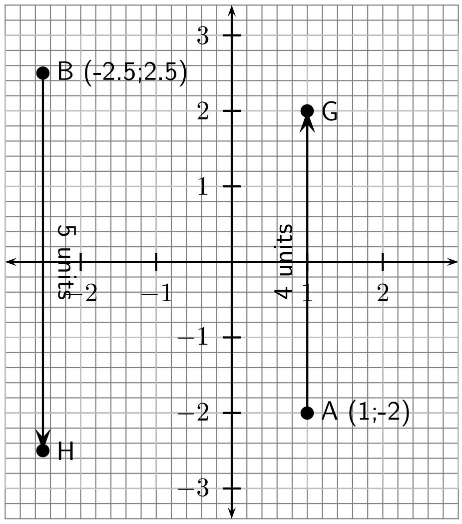
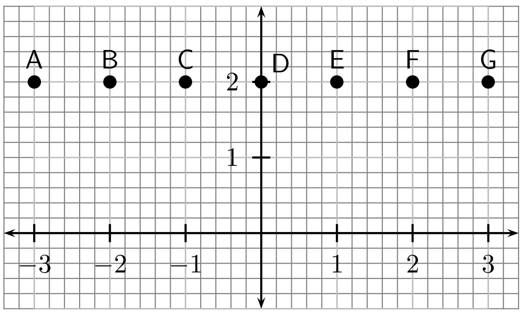


Figure 13.3: Point A is moved 4 units upwards to the position marked by G. Point B is 5 units downwards to the position marked by H.



Point	x co-ordinate	y co-ordinate
A		
B		
C		
D		
E		
F		
G		

What do you notice about the x co-ordinates? What do you notice about the y co-ordinates?

What would happen to the co-ordinates of point A, if it was moved to the position of point G?

When a point is moved horizontally left or right on the Cartesian plane, the y co-ordinate of the point remains the same, but the x co-ordinate changes by the amount that the point was moved left or right.

For example, in Figure 13.4 Point A is moved 4 units right to the position marked by G. The new y co-ordinate of point A is the same (y=1), but the new x co-ordinate is shifted in the positive x direction 4 units and becomes x=-2**+4**=2. The new co-ordinate of point A at G is therefore (2;1). Similarly, for point B that is moved left by 5 units, the y co-ordinate is the same ($y = -2{,}5$), but the x co-ordinate is shifted in the negative x-direction by 5 units. The new x co-ordinate is therefore x=2,5 **-5**=-2,5. The new co-ordinates of point B at H is therefore (-2,5;1).

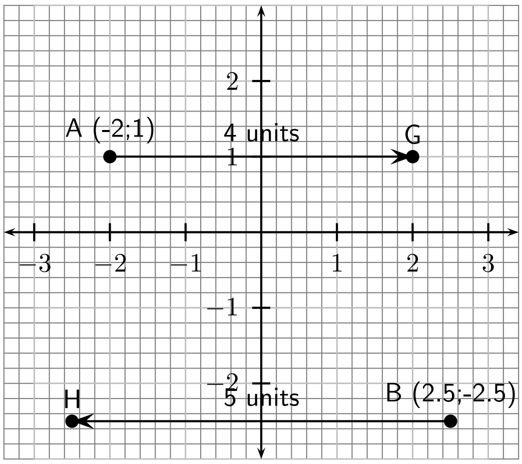
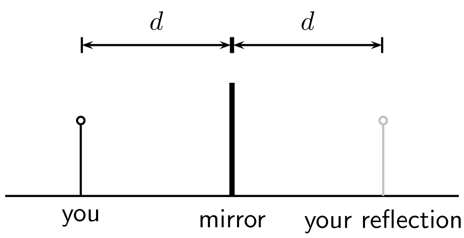


Figure 13.4: Point A is moved 4 units to the right to the position marked by G. Point B is 5 units to the left to the position marked by H.

Important: If a point is shifted to the right, the new x co-ordinate is given by adding the shift to the old x co-ordinate. If a point is shifted to the left, the new x co-ordinate is given by subtracting the shift from the old x co-ordinate.

13.5.2 Reflection of a Point

When you stand in front of a mirror your reflection is located the same distance (d) behind the mirror as you are standing in front of the mirror.



We can apply the same idea to a point that is reflected on the x-axis, the y-axis and the line $y = x$.

Reflection on the x-axis

If a point is reflected on the x-axis, then the reflection must be the same distance below the x-axis as the point is above the x-axis and vice-versa.

Important: When a point is reflected about the x-axis, only the y co-ordinate of the point changes.

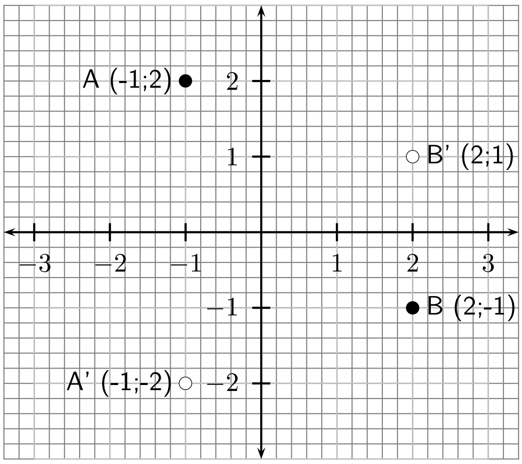


Figure 13.5: Points A and B are reflected on the x-axis. The original points are shown with • and the reflected points are shown with ◦.

Worked Example 53: Reflection on the x-axis

Question: Find the co-ordinates of the reflection of the point P, if P is reflected on the x-axis. The co-ordinates of P are (5;10).
Answer
Step 1 : Determine what is given and what is required
We are given the point P with co-ordinates (5;10) and need to find the co-ordinates of the point if it is reflected on the x-axis.
Step 2 : Determine how to approach the problem
The point P is above the x-axis, therefore its reflection will be the same distance below the x-axis as the point P is above the x-axis. Therefore, y=-10.
For a reflection on the x-axis, the x co-ordinate remains unchanged. Therefore, x=5.
Step 3 : Write the final answer
The co-ordinates of the reflected point are (5;-10).

Reflection on the y-axis

If a point is reflected on the y-axis, then the reflection must be the same distance to the left of the y-axis as the point is to the right of the y-axis and vice-versa.

Important: When a point is reflected on the y-axis, only the x co-ordinate of the point changes. The y co-ordinate remains unchanged.

Worked Example 54: Reflection on the y-axis

Question: Find the co-ordinates of the reflection of the point Q, if Q is reflected on the y-axis. The co-ordinates of Q are (15;5).
Answer
Step 1 : Determine what is given and what is required

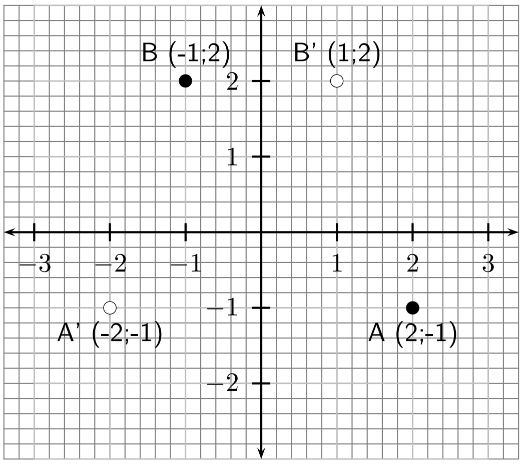


Figure 13.6: Points A and B are reflected on the y-axis. The original points are shown with • and the reflected points are shown with ∘.

We are given the point Q with co-ordinates (15;5) and need to find the co-ordinates of the point if it is reflected on the y-axis.

Step 2 : Determine how to approach the problem

The point Q is to the right of the y-axis, therefore its reflection will be the same distance to the left of the y-axis as the point Q is to the right of the y-axis. Therefore, x=-15.

For a reflection on the y-axis, the y co-ordinate remains unchanged. Therefore, y=5.

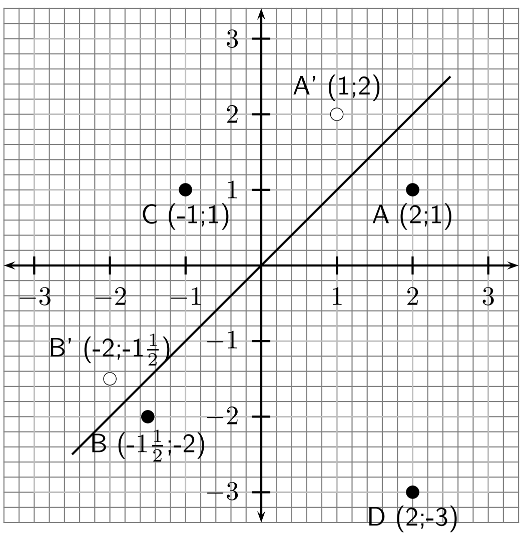
Step 3 : Write the final answer

The co-ordinates of the reflected point are (-15;5).

Reflection on the line $y = x$

The final type of reflection you will learn about is the reflection of a point on the line $y = x$.

Activity :: Casestudy : Reflection of a point on the line $y = x$



Study the information given and complete the following table:

	Point	Reflection
A	(2;1)	(1;2)
B	(-1$\frac{1}{2}$;-2)	(-2;-1$\frac{1}{2}$)
C	(-1;1)	
D	(2;-3)	

What can you deduce about the co-ordinates of points that are reflected about the line $y = x$?

The x and y co-ordinates of points that are reflected on the line $y = x$ are swapped around, or interchanged. This means that the x co-ordinate of the original point becomes the y co-ordinate of the reflected point and the y co-ordinate of the original point becomes the x co-ordinate of the reflected point.

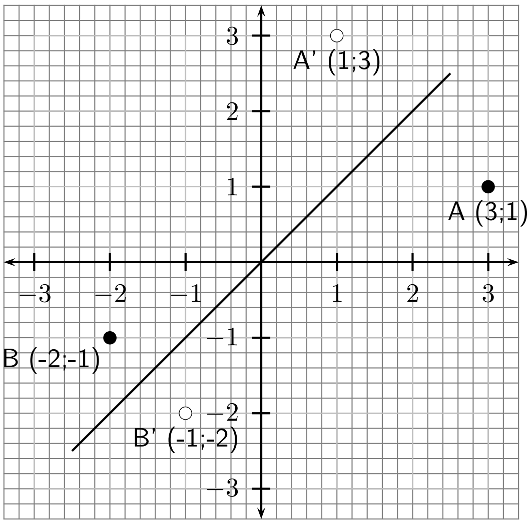


Figure 13.7: Points A and B are reflected on the line $y = x$. The original points are shown with • and the reflected points are shown with ◦.

Important: The x and y co-ordinates of points that are reflected on the line $y = x$ are interchanged.

Worked Example 55: Reflection on the line $y = x$

Question: Find the co-ordinates of the reflection of the point R, if R is reflected on the line $y = x$. The co-ordinates of R are (-5;5).
Answer
Step 1 : Determine what is given and what is required
We are given the point R with co-ordinates (-5;5) and need to find the co-ordinates of the point if it is reflected on the line $y = x$.
Step 2 : Determine how to approach the problem
The x co-ordinate of the reflected point is the y co-ordinate of the original point. Therefore, x=5.
The y co-ordinate of the reflected point is the x co-ordinate of the original point. Therefore, y=-5.
Step 3 : Write the final answer
The co-ordinates of the reflected point are (5;-5).

Rules of Translation

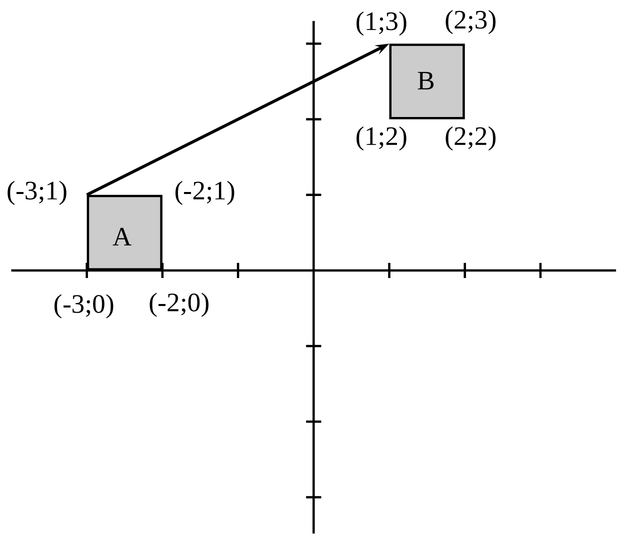
A quick way to write a translation is to use a 'rule of translation'. For example $(x;y) \rightarrow (x+a;y+b)$ means translate point (x;y) by moving a units horizontally and b units vertically. So if we translate (1;2) by the rule $(x;y) \rightarrow (x+3;y-1)$ it becomes (4;1). We have moved 3 units right and 1 unit down.

Translating a Region

To translate a region, we translate each point in the region.

Example

Region A has been translated to region B by the rule: $(x;y) \rightarrow (x+4;y+2)$



Activity :: Discussion : Rules of Transformations

Work with a friend and decide which item from column 1 matches each description in column 2.

Column 1	Column 2
$(x;y) \rightarrow (x;y-3)$	a reflection on x-y line
$(x;y) \rightarrow (x-3;y)$	a reflection on the x axis
$(x;y) \rightarrow (x;-y)$	a shift of 3 units left
$(x;y) \rightarrow (-x;y)$	a shift of 3 units down
$(x;y) \rightarrow (y;x)$	a reflection on the y-axis

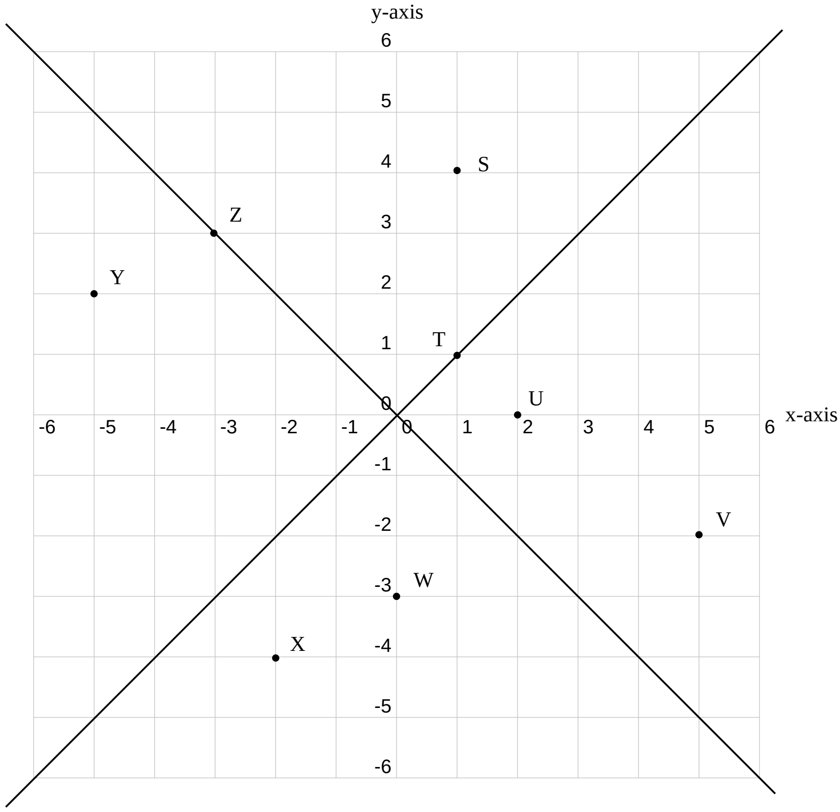
?

Exercise: Transformations

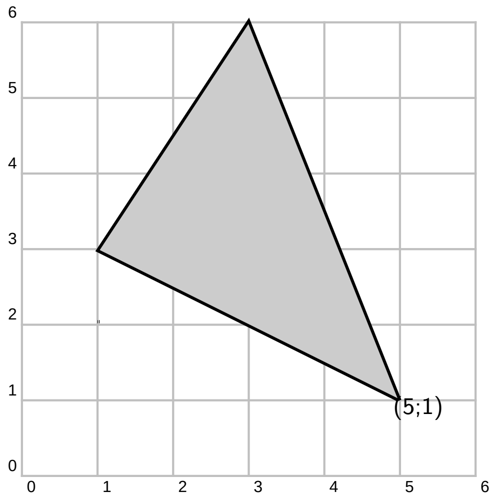
1. Find the co-ordinates of each of the points (S - Z) if they are reflected about the given lines:

   a) y-axis (x=0)

b) x-axis (y=0)
c) y=-x
d) y=x



2. Write down the rule used for each of the following reflections:
   a) Z(7;3), Z'(3;7)
   b) Y(-1;-8), Y'(1;-8)
   c) X(5;9), X'(-5;9)
   d) W(4;6), W'(4;6)
   e) V($\frac{-3}{7}$;$\frac{5}{3}$), V'($\frac{5}{3}$;$\frac{-3}{7}$)
3. a) Reflect the given points using the rules that are given.
   b) Identify the line of reflection in each case (some may not exist):
      (i) H(-4;3); (x;y)→ (-x;y)
      (ii) H(-4;3); (x;y) → (-y;-x)
      (iii) H(-4;3); (x;y) → (y;x)
      (iv) H(-4;3); (x;y) → (-x;-y)
      (v) H(-4;3); (x;y) → (x;-y)



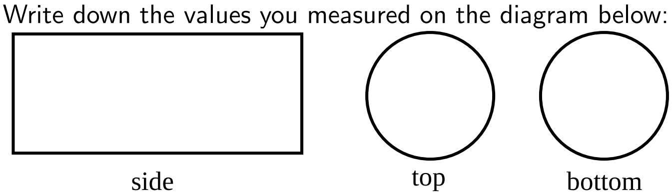
4. Using squared paper, copy the diagram given. Let -10≤x≤10, -10≤y≤10.

i) Identify the transformation.
ii) Draw the image of the figure according the rules given.

a) $(x; y) \rightarrow (-x; y)$
b) $(x; y) \rightarrow (y; x)$
c) $(x; y) \rightarrow (x; y - 3)$
d) $(x; y) \rightarrow (x + 5; y)$
e) $(x; y) \rightarrow (x; -y)$

Activity :: Investigation : Calculation of Volume, Surface Area and scale factors of objects

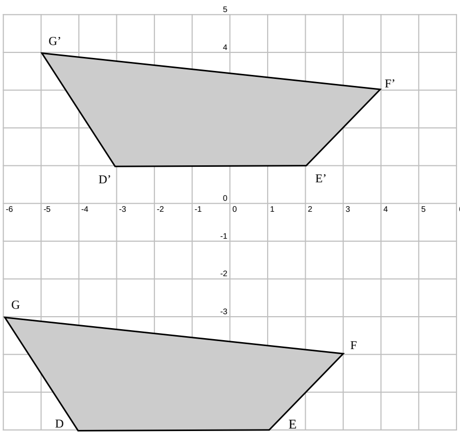
1. Look around the house or school and find a can or a tin of any kind (e.g. beans, soup, cooldrink, paint etc.)
2. Measure the height of the tin and the diameter of its top or bottom.
3. Write down the values you measured on the diagram below:

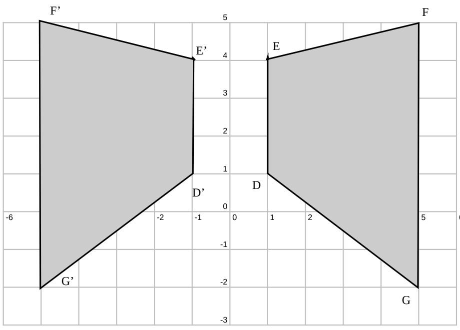


4. Using your measurements, calculate the following (in cm^2, rounded off to 2 decimal places):
   (a) the area of the side of the tin (i.e. the rectangle)
   (b) the area of the top and bottom of the tin (i.e. the circles)
   (c) the total surface area of the tin
5. If the tin metal costs 0,17 cents/cm^2, how much does it cost to make the tin?
6. Find the volume of your tin (in cm^3, rounded off to 2 decimal places).
7. What is the volume of the tin given on its label?
8. Compare the volume you calculated with the value given on the label. How much air is contained in the tin when it contains the product (i.e. cooldrink, soup etc.)
9. Why do you think space is left for air in the tin?
10. If you wanted to double the volume of the tin, but keep the radius the same, by how much would you need to increase the height?
11. If the height of the tin is kept the same, but now the radius is doubled, by what scale factor will the:
    (a) area of the side surface of the tin increase?
    (b) area of the bottom/top of the tin increase?

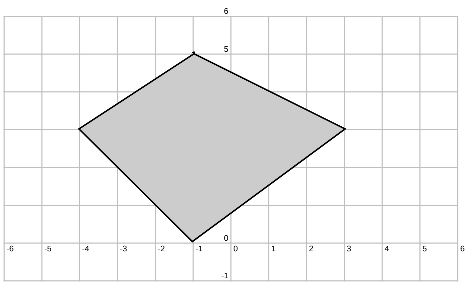
13.6 End of Chapter Exercises

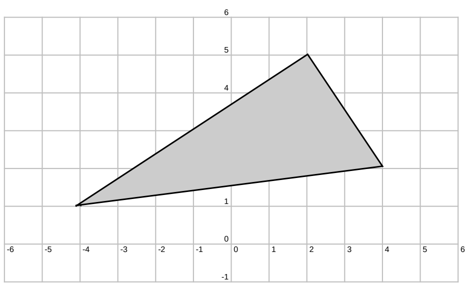
1. Write a rule that will give the following transformations of DEFG to D'E'F'G in each case.

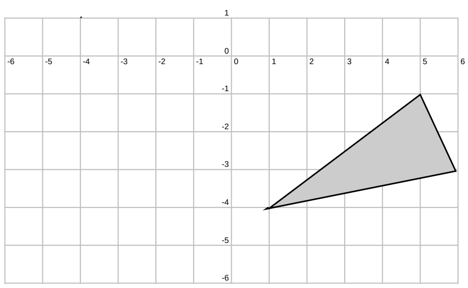
a) 

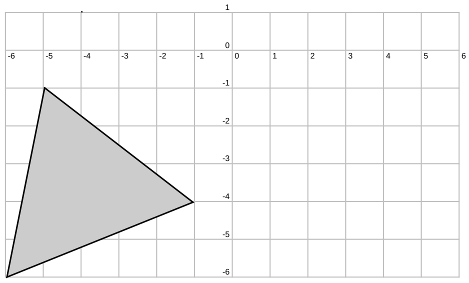
b) 

2. Using the rules given, identify the type of transformation and draw the image of the shapes.

a) (x;y)→(x+3;y-3) 

b) (x;y)→(x-4;y) 

c) (x;y)→(y;x) 

d) (x;y)→(-x;-y) 

Appendix A

GNU Free Documentation License

PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or non-commercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.

A section "Entitled XYZ" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "Acknowledgements", "Dedications", "Endorsements", or "History".) To "Preserve the Title" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or non-commercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section A.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections A and A above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

1. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
2. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
3. State on the Title page the name of the publisher of the Modified Version, as the publisher.
4. Preserve all the copyright notices of the Document.
5. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
6. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
7. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
8. Include an unaltered copy of this License.
9. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

10. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.

11. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

12. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

13. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version.

14. Do not re-title any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section.

15. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties–for example, statements of peer review or that the text has been approved by an organisation as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section A above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements".

COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section A is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section A. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section A) to Preserve its Title (section A) will typically require changing the actual title.

TERMINATION

You may not copy, modify, sub-license, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sub-license or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

> Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the "with...Texts." line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.