

FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

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Chapter 13

Geometry - Grade 10

13.1 Introduction

Geometry (Greek: geo = earth, metria = measure) arose as the field of knowledge dealing with spatial relationships. It was one of the two fields of pre-modern mathematics, the other being the study of numbers. In modern times, geometric concepts have become very complex and abstract and are barely recognizable as the descendants of early geometry.

Activity :: Researchproject : History of Geometry

Work in pairs or groups and investigate the history of the foundation of geometry. Describe the various stages of development and how the following cultures used geometry to improve their lives.

1. Ancient Indian geometry (c. 3000 - 500 B.C.)
 - (a) Harappan geometry
 - (b) Vedic geometry
 2. Classical Greek geometry (c. 600 - 300 B.C.)
 - (a) Thales and Pythagoras
 - (b) Plato
 3. Hellenistic geometry (c. 300 B.C - 500 C.E.)
 - (a) Euclid
 - (b) Archimedes
-

13.2 Right Prisms and Cylinders

In this section we study how to calculate the surface areas and volumes of right prisms and cylinders. A right prism is a polygon that has been stretched out into a tube so that the height of the tube is perpendicular to the base. A square prism has a base that is a square and a triangular prism has a base that is a triangle.

It is relatively simple to calculate the surface areas and volumes of prisms.

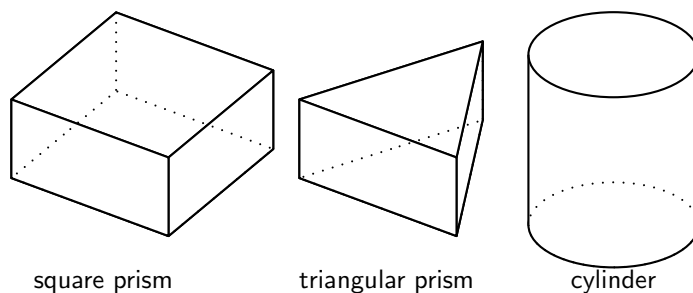


Figure 13.1: Examples of a right square prism, a right triangular prism and a cylinder.

13.2.1 Surface Area

The term *surface area* refers to the total area of the exposed or outside surfaces of a prism. This is easier to understand if you imagine the prism as a solid object.

If you examine the prisms in Figure 13.1, you will see that each face of a prism is a simple polygon. For example, the triangular prism has two faces that are triangles and three faces that are rectangles. Therefore, in order to calculate the surface area of a prism you simply have to calculate the area of each face and add it up. In the case of a cylinder the top and bottom faces are circles, while the curved surface flattens into a rectangle.

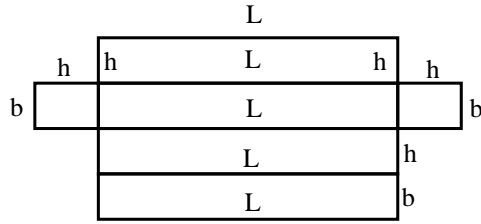
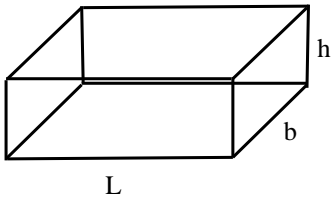
Surface Area of Prisms

Calculate the area of each face and add the areas together to get the surface area.

Activity :: Discussion : surface areas

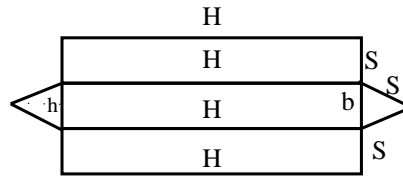
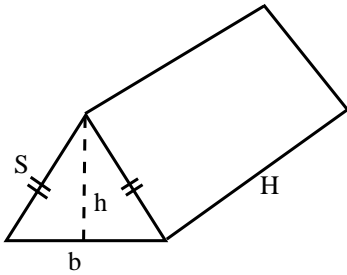
Study the following prisms, nets and formulae. Explain to your partner, how each relates to the other.

Rectangular Prism



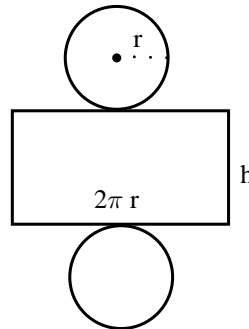
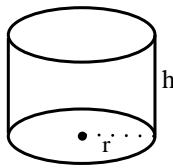
$$S.A. = 2[(L \times b) + (b \times h) + (L \times h)]$$

Triangular Prism

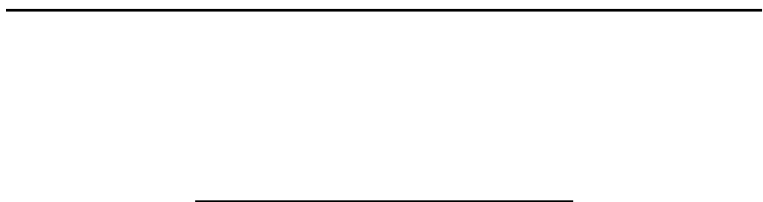


$$S.A. = 2\left(\frac{1}{2}b \times h\right) + 2(H \times S) + (H \times b)$$

Cylinder



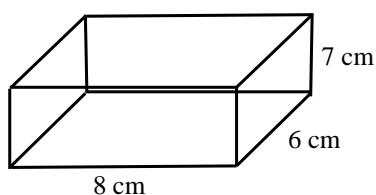
$$S.A. = 2\pi r^2 + 2\pi r h$$



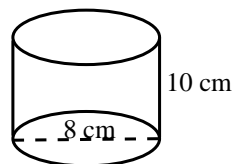
Exercise: Surface areas

1. Calculate the surface area in each of the following:

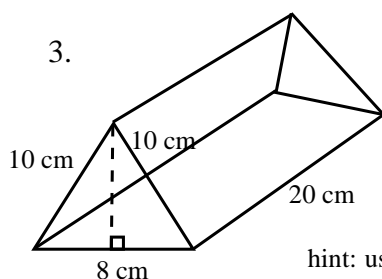
1.



2.

hint: diameter = $2 \times$ radius

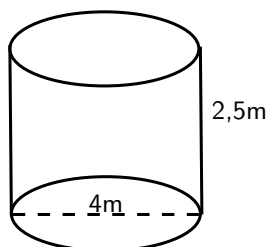
3.



hint: use Pythagoras to find height of triangular face.

2. If a litre of paint, paints $2m^2$, how much paint is needed to paint:

- A rectangular swimming pool with dimensions $4m \times 3m \times 2,5m$, inside walls and floor only.
- The inside walls and floor of a circular reservoir with diameter $4m$ and height $2,5m$



13.2.2 Volume

The volume of a right prism is calculated by multiplying the area of the base by the height. So, for a square prism of side length a and height h the volume is $a \times a \times h = a^2h$.

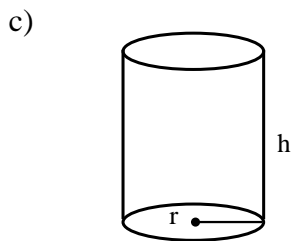
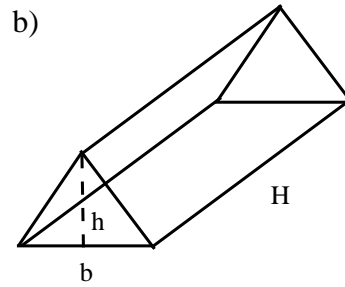
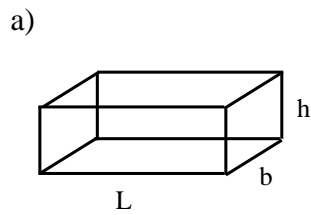
Volume of Prisms

Calculate the area of the base and multiply by the height to get the volume of a prism.

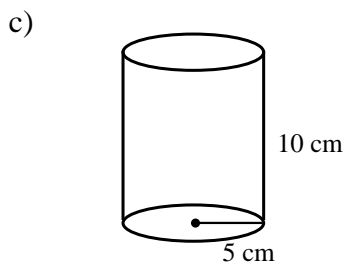
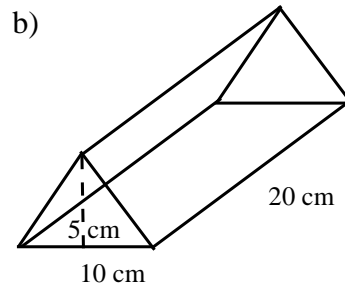
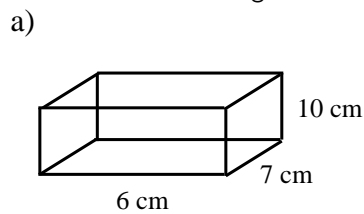


Exercise: Volume

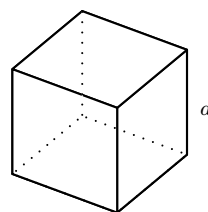
- Write down the formula for each of the following volumes:



2. Calculate the following volumes:



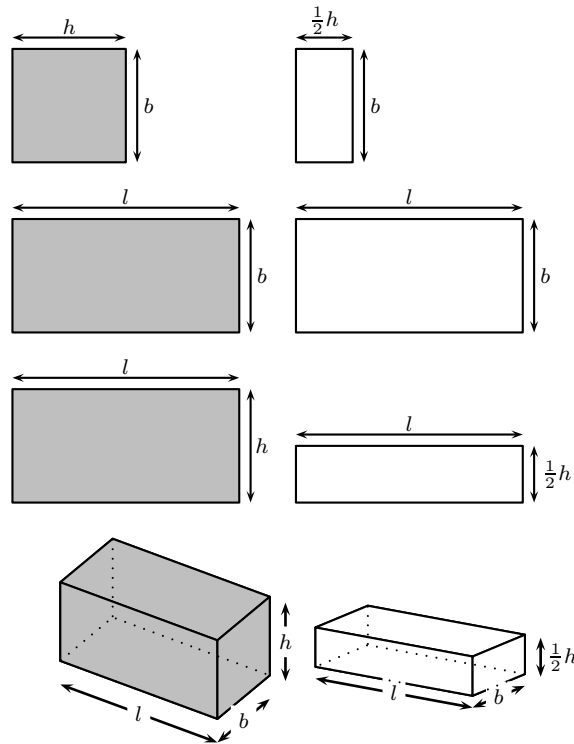
3. A cube is a special prism that has all edges equal. This means that each face is a square. An example of a cube is a die. Show that for a cube with side length a , the surface area is $6a^2$ and the volume is a^3 .



Now, what happens to the surface area if one dimension is multiplied by a constant? For example, how does the surface area change when the height of a rectangular prism is divided by 2?



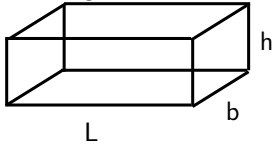
Worked Example 50: Scaling the dimensions of a prism



Surface Area = $2(l \times h + l \times b + b \times h)$	Surface Area = $2(l \times \frac{1}{2}h + l \times b + b \times \frac{1}{2}h)$
Volume = $l \times b \times h$	Volume = $l \times b \times \frac{1}{2}h$ = $\frac{1}{2}(l \times b \times h)$

Figure 13.2: Rectangular prisms

Question: The size of a prism is specified by the length of its sides. The prism in the diagram has sides of lengths L , b and h .



- a) Consider enlarging all sides of the prism by a constant factor x . Where $x > 1$. Calculate the volume and surface area of the enlarged prism as a function of the factor x and the volume of the original volume.
- a) In the same way as above now consider the case, where $0 < x < 1$. Now calculate the reduction factor in the volume and the surface area.

Answer

Step 1 : Identify

The volume of a prism is given by:

$$V = L \times b \times h$$

The surface area of the prism is given by:

$$A = 2 \times (L \times b + L \times h + b \times h)$$

Step 2 : Rescale

If all the sides of the prism get rescaled, the new sides will be:

$$L' = x \times L$$

$$b' = x \times b$$

$$h' = x \times h$$

The new volume will then be given by:

$$\begin{aligned}
 V' &= L' \times b' \times h' \\
 &= x \times L \times x \times b \times x \times h \\
 &= x^3 \times L \times b \times h \\
 &= x^3 \times V
 \end{aligned}$$

The new surface area of the prism will be given by:

$$\begin{aligned}
 A' &= 2 \times (L' \times b' + L' \times h' + b' \times h') \\
 &= 2 \times (x \times L \times x \times b + x \times L \times x \times h + x \times b \times x \times h) \\
 &= x^2 \times 2 \times (L \times b + L \times h + b \times h) \\
 &= x^2 \times A
 \end{aligned}$$

Step 3 : Interpret

a) We found above that the new volume is given by:

$$V' = x^3 \times V$$

Since $x > 1$, the volume of the prism will be increased by a factor of x^3 .

The surface area of the rescaled prism was given by:

$$A' = x^2 \times A$$

Again, since $x > 1$, the surface area will be increased by a factor of x^2 .

b) The answer here is based on the same ideas as above.

In analogy, since here $0 < x < 1$, the volume will be reduced by a factor of x^3 and the surface area will be decreased by a factor of x^2 .

When the length of one of the sides is multiplied by a constant the effect is to multiply the original volume by that constant, as for the example in Figure 13.2.

13.3 Polygons

Polygons are all around us. A stop sign is in the shape of an octagon, an eight-sided polygon. The honeycomb of a beehive consists of hexagonal cells.

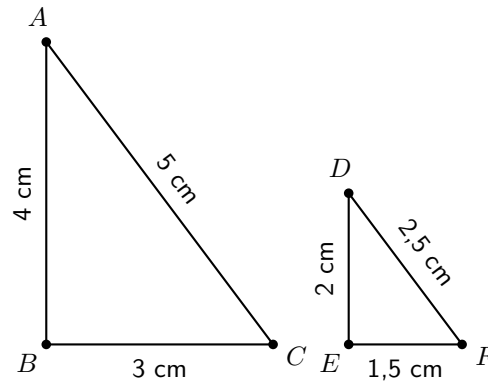
In this section, you will learn about similar polygons.

13.3.1 Similarity of Polygons

Activity :: Discussion : Similar Triangles

Fill in the table using the diagram and then answer the questions that follow.

$\frac{AB}{DE} = \frac{\dots cm}{\dots cm} = \dots$	$\hat{A} = \dots^\circ$	$\hat{D} = \dots^\circ$
$\frac{BC}{EF} = \frac{\dots cm}{\dots cm} = \dots$	$\hat{B} = \dots^\circ$	$\hat{E} = \dots^\circ$
$\frac{AC}{DF} = \frac{\dots cm}{\dots cm} = \dots$	$\hat{C} = \dots^\circ$	$\hat{F} = \dots^\circ$



1. What can you say about the numbers you calculated for: $\frac{AB}{DE}$, $\frac{BC}{EF}$, $\frac{AC}{DF}$?
2. What can you say about \hat{A} and \hat{D} ?
3. What can you say about \hat{B} and \hat{E} ?
4. What can you say about \hat{C} and \hat{F} ?

If two polygons are *similar*, one is an enlargement of the other. This means that the two polygons will have the same angles and their sides will be in the same proportion.

We use the symbol \cong to mean *is similar to*.



Definition: Similar Polygons

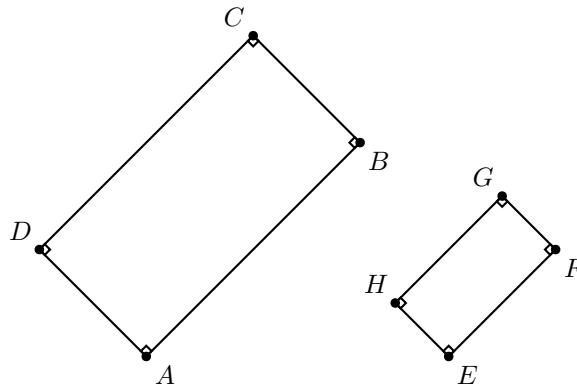
Two polygons are similar if:

1. their corresponding angles are equal, or
2. the ratios of corresponding sides are equal.



Worked Example 51: Similarity of Polygons

Question: Show that the following two polygons are similar.



Answer

Step 1 : Determine what is required

We are required to show that the pair of polygons is similar. We can do this by showing that the ratio of corresponding sides is equal or by showing that corresponding angles are equal.

Step 2 : Decide how to approach the problem

We are not given the lengths of the sides, but we are given the angles. So, we can show that corresponding angles are equal.

Step 3 : Show that corresponding angles are equal

All angles are given to be 90° and

$$\begin{aligned} \hat{A} &= \hat{E} \\ \hat{B} &= \hat{F} \\ \hat{C} &= \hat{G} \\ \hat{D} &= \hat{H} \end{aligned}$$

Step 4 : Final answer

Since corresponding angles are equal, the polygons ABCD and EFGH are similar.

Step 5 : Comment on result

This result shows that all rectangles are similar to each other, because all rectangles will always have corresponding angles equal to each other.

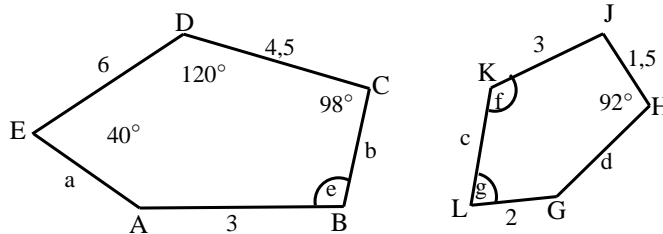


Important: All rectangles and squares are similar.



Worked Example 52: Similarity of Polygons

Question: If two pentagons ABCDE and GHJKL are similar, determine the lengths of the sides and angles labelled with letters:



Answer

Step 1 : Determine what is given

We are given that ABCDE and GHJKL are similar. This means that:

$$\frac{AB}{GH} = \frac{BC}{HJ} = \frac{CD}{JK} = \frac{DE}{KL} = \frac{EA}{LG}$$

and

$$\begin{aligned} \hat{A} &= \hat{G} \\ \hat{B} &= \hat{H} \\ \hat{C} &= \hat{J} \\ \hat{D} &= \hat{K} \\ \hat{E} &= \hat{L} \end{aligned}$$

Step 2 : Determine what is required

We are required to determine the following lengths:

1. a , b , c and d

and the following angles:

1. e , f and g

Step 3 : Decide how to approach the problem

The corresponding angles are equal, so no calculation is needed. We are given one pair of sides DC and KJ that correspond $\frac{DC}{KJ} = \frac{4,5}{3} = 1,5$ so we know that all sides of $KJHGL$ are 1,5 times smaller than $ABCDE$.

Step 4 : Calculate lengths

$$\begin{aligned}\frac{a}{2} &= 1,5 \quad \therefore a = 2 \times 1,5 = 3 \\ \frac{b}{1,5} &= 1,5 \quad \therefore b = 1,5 \times 1,5 = 2,25 \\ \frac{6}{c} &= 1,5 \quad \therefore c = 6 \div 1,5 = 4 \\ \frac{3}{d} &= 1,5 \quad \therefore d = 2\end{aligned}$$

Step 5 : Calculate angles

$$\begin{aligned}e &= 92^\circ \text{ (corresponds to H)} \\ f &= 120^\circ \text{ (corresponds to D)} \\ g &= 40^\circ \text{ (corresponds to E)}\end{aligned}$$

Step 6 : Write the final answer

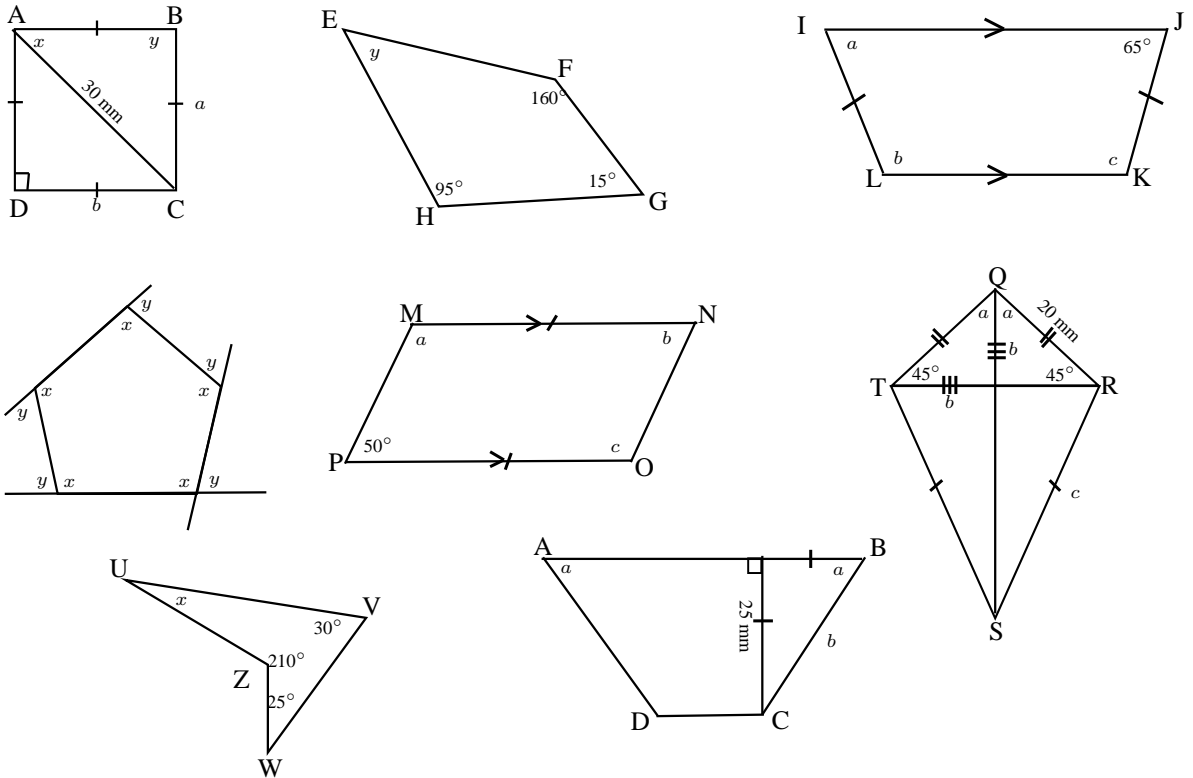
$$\begin{aligned}a &= 3 \\ b &= 2,25 \\ c &= 4 \\ d &= 2 \\ e &= 92^\circ \\ f &= 120^\circ \\ g &= 40^\circ\end{aligned}$$

Activity :: Similarity of Equilateral Triangles : Working in pairs, show that all equilateral triangles are similar.

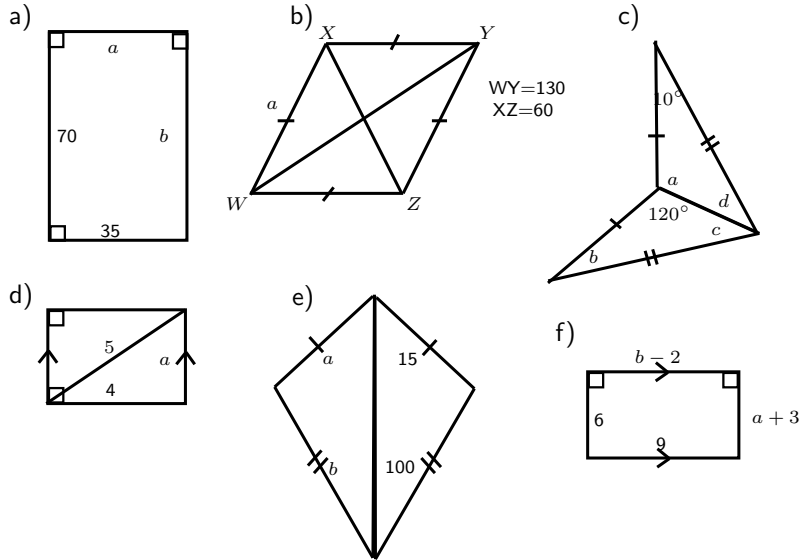
Exercise: Polygons-mixed

1. Find the values of the unknowns in each case. Give reasons.





2. Find the angles and lengths marked with letters in the following figures:



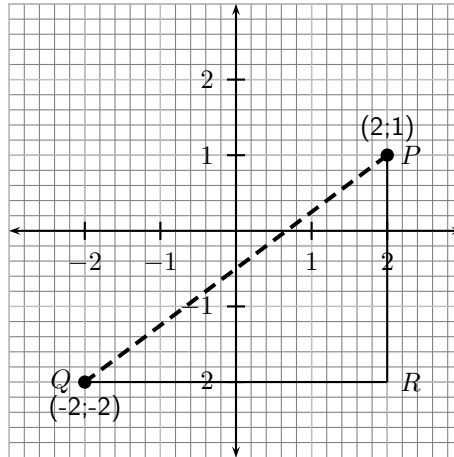
13.4 Co-ordinate Geometry

13.4.1 Introduction

Analytical geometry, also called co-ordinate geometry and earlier referred to as Cartesian geometry, is the study of geometry using the principles of algebra, and the Cartesian co-ordinate system. It is concerned with defining geometrical shapes in a numerical way, and extracting numerical information from that representation. Some consider that the introduction of analytic geometry was the beginning of modern mathematics.

13.4.2 Distance between Two Points

One of the simplest things that can be done with analytical geometry is to calculate the distance between two points. *Distance* is a number that describes how far apart two points are. For example, point P has co-ordinates $(2; 1)$ and point Q has co-ordinates $(-2; -2)$. How far apart are points A and B ? In the figure, this means how long is the dashed line?

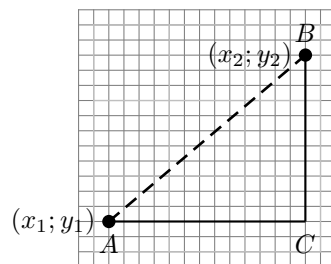


In the figure, it can be seen that the length of the line PR is 3 units and the length of the line QR is four units. However, the $\triangle PQR$, has a right angle at R . Therefore, the length of the side PQ can be obtained by using the Theorem of Pythagoras:

$$\begin{aligned} PQ^2 &= PR^2 + QR^2 \\ \therefore PQ^2 &= 3^2 + 4^2 \\ \therefore PQ &= \sqrt{3^2 + 4^2} = 5 \end{aligned}$$

The length of AB is the distance between the points A and B .

In order to generalise the idea, assume A is any point with co-ordinates $(x_1; y_1)$ and B is any other point with co-ordinates $(x_2; y_2)$.



The formula for calculating the distance between two points is derived as follows. The distance between the points A and B is the length of the line AB . According to the Theorem of Pythagoras, the length of AB is given by:

$$AB = \sqrt{AC^2 + BC^2}$$

However,

$$\begin{aligned} BC &= y_2 - y_1 \\ AC &= x_2 - x_1 \end{aligned}$$

Therefore,

$$\begin{aligned} AB &= \sqrt{AC^2 + BC^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

Therefore, for any two points, $(x_1; y_1)$ and $(x_2; y_2)$, the formula is:

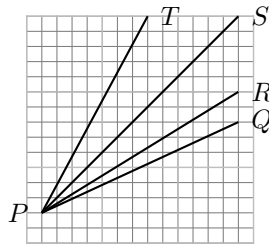
$$\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Using the formula, distance between the points P and Q with co-ordinates $(2;1)$ and $(-2;-2)$ is then found as follows. Let the co-ordinates of point P be $(x_1; y_1)$ and the co-ordinates of point Q be $(x_2; y_2)$. Then the distance is:

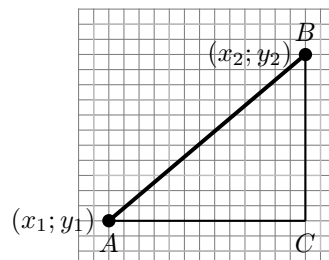
$$\begin{aligned} \text{Distance} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2 - (-2))^2 + (1 - (-2))^2} \\ &= \sqrt{(2 + 2)^2 + (1 + 2)^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

13.4.3 Calculation of the Gradient of a Line

The gradient of a line describes how steep the line is. In the figure, line PT is the steepest. Line PS is less steep than PT but is steeper than PR , and line PR is steeper than PQ .



The gradient of a line is defined as the ratio of the vertical distance to the horizontal distance. This can be understood by looking at the line as the hypotenuse of a right-angled triangle. Then the gradient is the ratio of the length of the vertical side of the triangle to the horizontal side of the triangle. Consider a line between a point A with co-ordinates $(x_1; y_1)$ and a point B with co-ordinates $(x_2; y_2)$.



$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

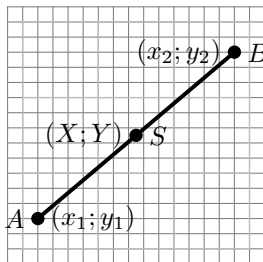
For example the gradient of the line between the points P and Q , with co-ordinates $(2;1)$ and $(-2;-2)$ (Figure 13.4.2) is:

$$\begin{aligned}
 \text{Gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{-2 - 1}{-2 - 2} \\
 &= \frac{-3}{-4} \\
 &= \frac{3}{4}
 \end{aligned}$$

13.4.4 Midpoint of a Line

Sometimes, knowing the co-ordinates of the middle point or *midpoint* of a line is useful. For example, what is the midpoint of the line between point P with co-ordinates $(2; 1)$ and point Q with co-ordinates $(-2; -2)$.

The co-ordinates of the midpoint of any line between any two points A and B with co-ordinates $(x_1; y_1)$ and $(x_2; y_2)$, is generally calculated as follows. Let the midpoint of AB be at point S with co-ordinates $(X; Y)$. The aim is to calculate X and Y in terms of $(x_1; y_1)$ and $(x_2; y_2)$.



$$\begin{aligned}
 X &= \frac{x_1 + x_2}{2} \\
 Y &= \frac{y_1 + y_2}{2} \\
 \therefore S &\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)
 \end{aligned}$$

Then the co-ordinates of the midpoint (S) of the line between point P with co-ordinates $(2; 1)$ and point Q with co-ordinates $(-2; -2)$ is:

$$\begin{aligned}
 X &= \frac{x_1 + x_2}{2} \\
 &= \frac{-2 + 2}{2} \\
 &= 0 \\
 Y &= \frac{y_1 + y_2}{2} \\
 &= \frac{-2 + 1}{2} \\
 &= -\frac{1}{2} \\
 \therefore S &\left(0; -\frac{1}{2} \right)
 \end{aligned}$$

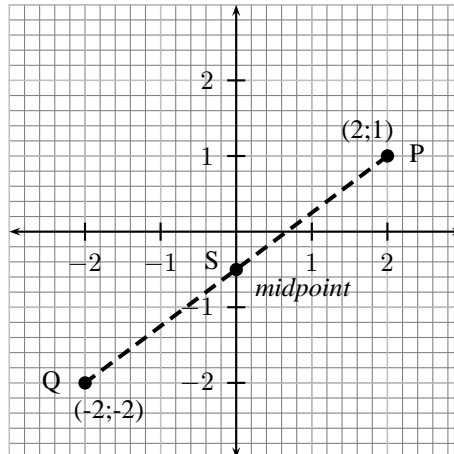
It can be confirmed that the distance from the each end point to the midpoint is equal. The co-ordinate of the midpoint S is $(0; -0,5)$.

$$\begin{aligned}
 PS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
 &= \sqrt{(0 - 2)^2 + (-0.5 - 1)^2} \\
 &= \sqrt{(-2)^2 + (-1.5)^2} \\
 &= \sqrt{4 + 2.25} \\
 &= \sqrt{6.25}
 \end{aligned}$$

and

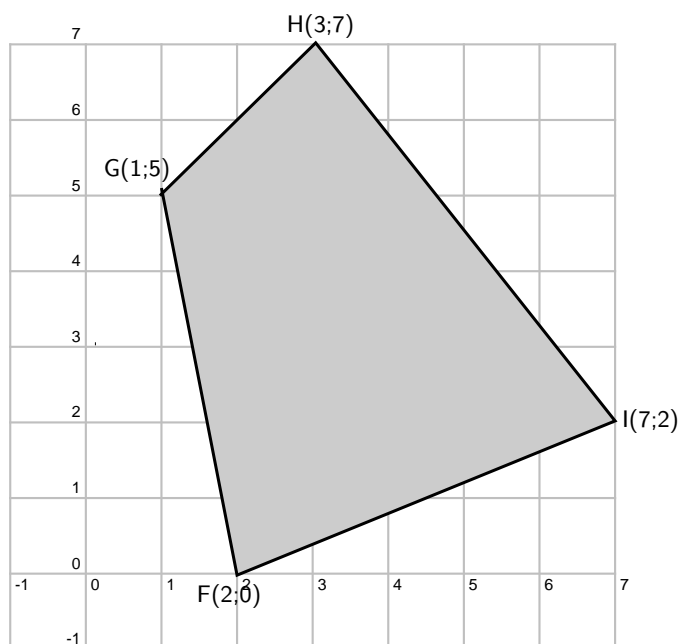
$$\begin{aligned}
 QS &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
 &= \sqrt{(0 - (-2))^2 + (-0.5 - (-2))^2} \\
 &= \sqrt{(0 + 2)^2 + (-0.5 + 2)^2} \\
 &= \sqrt{(2)^2 + (1.5)^2} \\
 &= \sqrt{4 + 2.25} \\
 &= \sqrt{6.25}
 \end{aligned}$$

It can be seen that $PS = QS$ as expected.



Exercise: Co-ordinate Geometry

1. In the diagram given the vertices of a quadrilateral are F(2;0), G(1;5), H(3;7)



and $I(7;2)$.

- a) What are the lengths of the opposite sides of FGHI?
 - b) Are the opposite sides of FGHI parallel?
 - c) Do the diagonals of FGHI bisect each other?
 - d) Can you state what type of quadrilateral FGHI is? Give reasons for your answer.
2. A quadrilateral ABCD with vertices $A(3;2)$, $B(1;7)$, $C(4;5)$ and $D(1;3)$ is given.
 - a) Draw the quadrilateral.
 - b) Find the lengths of the sides of the quadrilateral.
 3. $S(1;4)$, $T(-1;2)$, $U(0;-1)$ and $V(4;-1)$ are the vertices of a pentagon.
 - a) Are two of the sides of this pentagon parallel? If yes, find them.
 - b) Are two of the sides of this pentagon of equal length? If yes, find them.
 4. ABCD is a quadrilateral with vertices $A(0;3)$, $B(4;3)$, $C(5;-1)$ and $D(-1;-1)$.
 - a) Show that:
 - (i) $AD = BC$
 - (ii) $AB \parallel DC$
 - b) What name would you give to ABCD?
 - c) Show that the diagonals AC and BD do not bisect each other.
 5. P, Q, R and S are the points $(-2;0)$, $(2;3)$, $(5;3)$, $(-3;-3)$ respectively.
 - a) Show that:
 - (i) $SR = 2PQ$
 - (ii) $SR \parallel PQ$
 - b) Calculate:
 - (i) PS
 - (ii) QR
 - c) What kind of a quadrilateral is PQRS? Give reasons for your answers.
 6. EFGH is a parallelogram with vertices $E(-1;2)$, $F(-2;-1)$ and $G(2;0)$. Find the co-ordinates of H by using the fact that the diagonals of a parallelogram bisect each other.

13.5 Transformations

In this section you will learn about how the co-ordinates of a point change when the point is moved horizontally and vertically on the Cartesian plane. You will also learn about what happens to the co-ordinates of a point when it is reflected on the x -axis, y -axis and the line $y = x$.

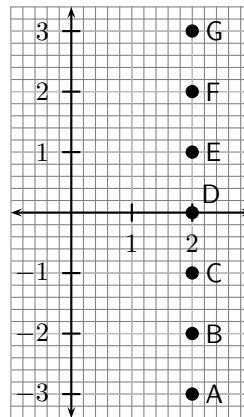
13.5.1 Translation of a Point

When something is moved in a straight line, we say that it is *translated*. What happens to the co-ordinates of a point that is translated horizontally or vertically?

Activity :: Discussion : Translation of a Point Vertically

Complete the table, by filling in the co-ordinates of the points shown in the figure.

Point	x co-ordinate	y co-ordinate
A		
B		
C		
D		
E		
F		
G		



What do you notice about the x co-ordinates? What do you notice about the y co-ordinates?

What would happen to the co-ordinates of point A, if it was moved to the position of point G?

When a point is moved vertically up or down on the Cartesian plane, the x co-ordinate of the point remains the same, but the y co-ordinate changes by the amount that the point was moved up or down.

For example, in Figure 13.3 Point A is moved 4 units upwards to the position marked by G. The new x co-ordinate of point A is the same ($x=1$), but the new y co-ordinate is shifted in the positive y direction 4 units and becomes $y=-2+4=2$. The new co-ordinates of point A are therefore G(1;2). Similarly, for point B that is moved downwards by 5 units, the x co-ordinate is the same ($x = -2,5$), but the y co-ordinate is shifted in the negative y -direction by 5 units. The new y co-ordinate is therefore $y=2,5 -5=-2,5$.



Important: If a point is shifted upwards, the new y co-ordinate is given by adding the shift to the old y co-ordinate. If a point is shifted downwards, the new y co-ordinate is given by subtracting the shift from the old y co-ordinate.

Activity :: Discussion : Translation of a Point Horizontally

Complete the table, by filling in the co-ordinates of the points shown in the figure.

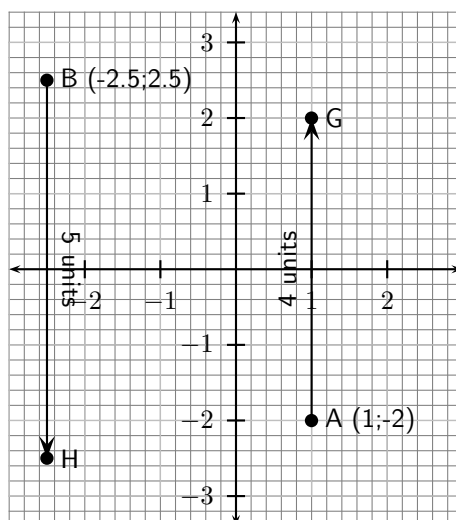
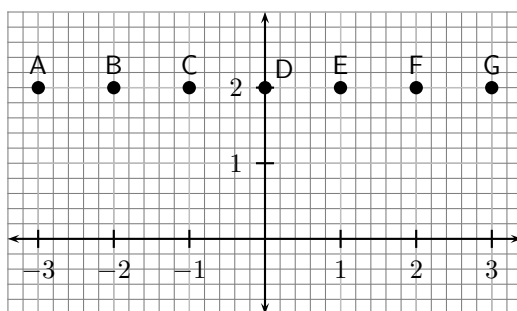


Figure 13.3: Point A is moved 4 units upwards to the position marked by G. Point B is 5 units downwards to the position marked by H.



Point	x co-ordinate	y co-ordinate
A		
B		
C		
D		
E		
F		
G		

What do you notice about the x co-ordinates? What do you notice about the y co-ordinates?

What would happen to the co-ordinates of point A, if it was moved to the position of point G?

When a point is moved horizontally left or right on the Cartesian plane, the y co-ordinate of the point remains the same, but the x co-ordinate changes by the amount that the point was moved left or right.

For example, in Figure 13.4 Point A is moved 4 units right to the position marked by G. The new y co-ordinate of point A is the same ($y=1$), but the new x co-ordinate is shifted in the positive x direction 4 units and becomes $x=-2+4=2$. The new co-ordinate of point A at G is therefore (2;1). Similarly, for point B that is moved left by 5 units, the y co-ordinate is the same ($y = -2,5$), but the x co-ordinate is shifted in the negative x -direction by 5 units. The new x co-ordinate is therefore $x=2,5 -5=-2,5$. The new co-ordinates of point B at H is therefore (-2,5;1).

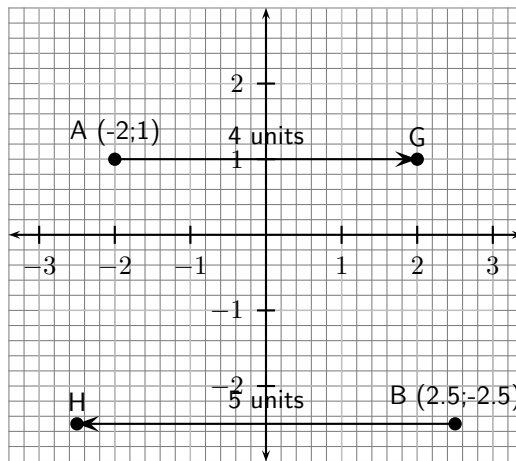


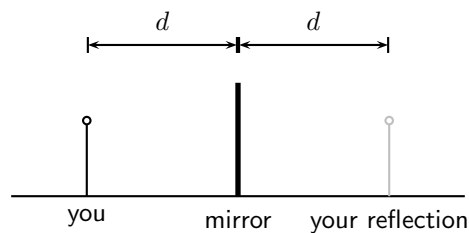
Figure 13.4: Point A is moved 4 units to the right to the position marked by G. Point B is 5 units to the left to the position marked by H.



Important: If a point is shifted to the right, the new x co-ordinate is given by adding the shift to the old x co-ordinate. If a point is shifted to the left, the new x co-ordinate is given by subtracting the shift from the old x co-ordinate.

13.5.2 Reflection of a Point

When you stand in front of a mirror your reflection is located the same distance (d) behind the mirror as you are standing in front of the mirror.



We can apply the same idea to a point that is reflected on the x -axis, the y -axis and the line $y = x$.

Reflection on the x -axis

If a point is reflected on the x -axis, then the reflection must be the same distance below the x -axis as the point is above the x -axis and vice-versa.



Important: When a point is reflected about the x -axis, only the y co-ordinate of the point changes.

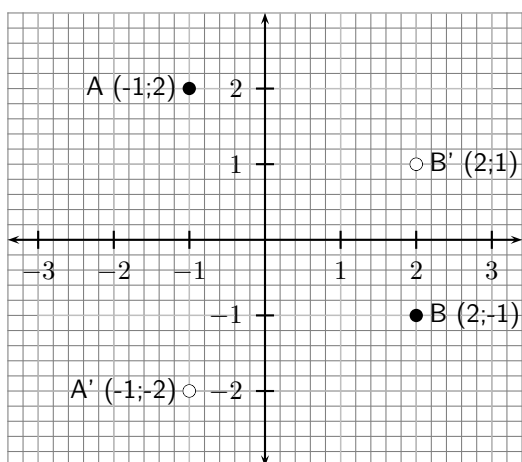


Figure 13.5: Points A and B are reflected on the x -axis. The original points are shown with \bullet and the reflected points are shown with \circ .



Worked Example 53: Reflection on the x -axis

Question: Find the co-ordinates of the reflection of the point P, if P is reflected on the x -axis. The co-ordinates of P are (5;10).

Answer

Step 1 : Determine what is given and what is required

We are given the point P with co-ordinates (5;10) and need to find the co-ordinates of the point if it is reflected on the x -axis.

Step 2 : Determine how to approach the problem

The point P is above the x -axis, therefore its reflection will be the same distance below the x -axis as the point P is above the x -axis. Therefore, $y=-10$.

For a reflection on the x -axis, the x co-ordinate remains unchanged. Therefore, $x=5$.

Step 3 : Write the final answer

The co-ordinates of the reflected point are (5;-10).

Reflection on the y -axis

If a point is reflected on the y -axis, then the reflection must be the same distance to the left of the y -axis as the point is to the right of the y -axis and vice-versa.



Important: When a point is reflected on the y -axis, only the x co-ordinate of the point changes. The y co-ordinate remains unchanged.



Worked Example 54: Reflection on the y -axis

Question: Find the co-ordinates of the reflection of the point Q, if Q is reflected on the y -axis. The co-ordinates of Q are (15;5).

Answer

Step 1 : Determine what is given and what is required

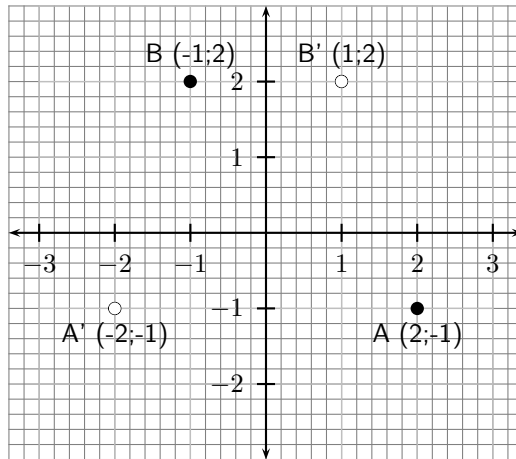


Figure 13.6: Points A and B are reflected on the y -axis. The original points are shown with \bullet and the reflected points are shown with \circ .

We are given the point Q with co-ordinates (15;5) and need to find the co-ordinates of the point if it is reflected on the y -axis.

Step 2 : Determine how to approach the problem

The point Q is to the right of the y -axis, therefore its reflection will be the same distance to the left of the y -axis as the point Q is to the right of the y -axis. Therefore, $x=-15$.

For a reflection on the y -axis, the y co-ordinate remains unchanged. Therefore, $y=5$.

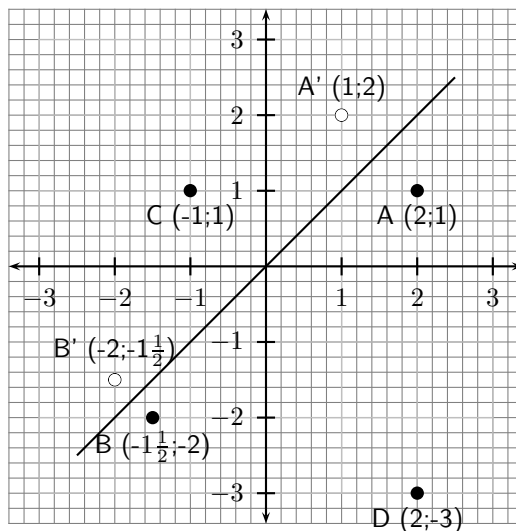
Step 3 : Write the final answer

The co-ordinates of the reflected point are (-15;5).

Reflection on the line $y = x$

The final type of reflection you will learn about is the reflection of a point on the line $y = x$.

Activity :: Casestudy : Reflection of a point on the line $y = x$



Study the information given and complete the following table:

	Point	Reflection
A	(2;1)	(1;2)
B	$(-1\frac{1}{2}; -2)$	$(-2; -1\frac{1}{2})$
C	(-1;1)	
D	(2;-3)	

What can you deduce about the co-ordinates of points that are reflected about the line $y = x$?

The x and y co-ordinates of points that are reflected on the line $y = x$ are swapped around, or interchanged. This means that the x co-ordinate of the original point becomes the y co-ordinate of the reflected point and the y co-ordinate of the original point becomes the x co-ordinate of the reflected point.

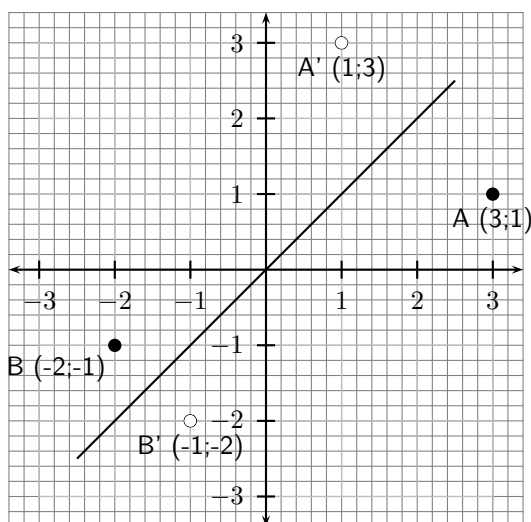


Figure 13.7: Points A and B are reflected on the line $y = x$. The original points are shown with \bullet and the reflected points are shown with \circ .

Important: The x and y co-ordinates of points that are reflected on the line $y = x$ are interchanged.

Worked Example 55: Reflection on the line $y = x$

Question: Find the co-ordinates of the reflection of the point R, if R is reflected on the line $y = x$. The co-ordinates of R are (-5;5).

Answer

Step 1 : Determine what is given and what is required

We are given the point R with co-ordinates (-5;5) and need to find the co-ordinates of the point if it is reflected on the line $y = x$.

Step 2 : Determine how to approach the problem

The x co-ordinate of the reflected point is the y co-ordinate of the original point. Therefore, $x=5$.

The y co-ordinate of the reflected point is the x co-ordinate of the original point. Therefore, $y=-5$.

Step 3 : Write the final answer

The co-ordinates of the reflected point are (5;-5).

Rules of Translation

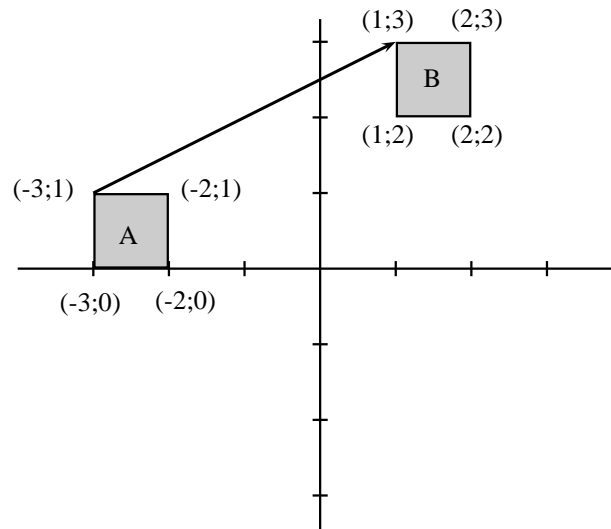
A quick way to write a translation is to use a 'rule of translation'. For example $(x; y) \rightarrow (x + a; y + b)$ means translate point $(x; y)$ by moving a units horizontally and b units vertically. So if we translate $(1; 2)$ by the rule $(x; y) \rightarrow (x + 3; y - 1)$ it becomes $(4; 1)$. We have moved 3 units right and 1 unit down.

Translating a Region

To translate a region, we translate each point in the region.

Example

Region A has been translated to region B by the rule: $(x; y) \rightarrow (x + 4; y + 2)$

**Activity :: Discussion : Rules of Transformations**

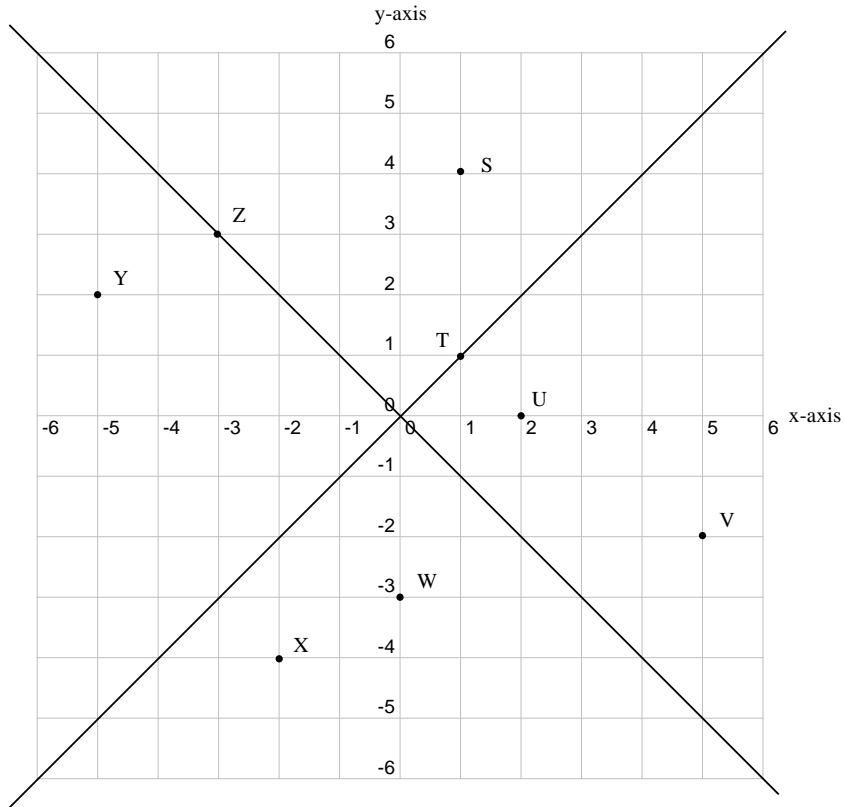
Work with a friend and decide which item from column 1 matches each description in column 2.

Column 1	Column 2
$(x; y) \rightarrow (x; y - 3)$	a reflection on x-y line
$(x; y) \rightarrow (x - 3; y)$	a reflection on the x axis
$(x; y) \rightarrow (x; -y)$	a shift of 3 units left
$(x; y) \rightarrow (-x; y)$	a shift of 3 units down
$(x; y) \rightarrow (y; x)$	a reflection on the y-axis

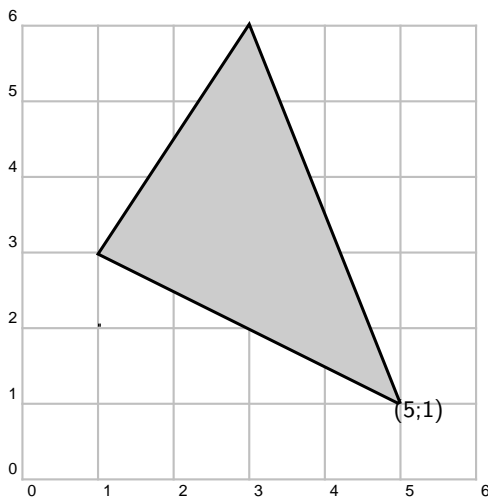
**Exercise: Transformations**

- Find the co-ordinates of each of the points (S - Z) if they are reflected about the given lines:
 - y-axis ($x=0$)

- b) x-axis ($y=0$)
- c) $y=-x$
- d) $y=x$



2. Write down the rule used for each of the following reflections:
 - a) $Z(7;3), Z'(3;7)$
 - b) $Y(-1;-8), Y'(1;-8)$
 - c) $X(5;9), X'(-5;9)$
 - d) $W(4;6), W'(4;6)$
 - e) $V(\frac{-3}{7};\frac{5}{3}), V'(\frac{5}{3};\frac{-3}{7})$
3. a) Reflect the given points using the rules that are given.
 b) Identify the line of reflection in each case (some may not exist):
 - (i) $H(-4;3); (x;y) \rightarrow (-x;y)$
 - (ii) $H(-4;3); (x;y) \rightarrow (-y;-x)$
 - (iii) $H(-4;3); (x;y) \rightarrow (y;x)$
 - (iv) $H(-4;3); (x;y) \rightarrow (-x;-y)$
 - (v) $H(-4;3); (x;y) \rightarrow (x;-y)$

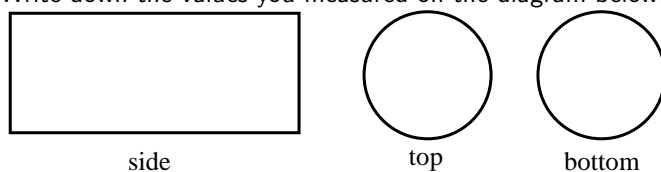


4. Using squared paper, copy the diagram given. Let $-10 \leq x \leq 10, -10 \leq y \leq 10$.

- i) Identify the transformation.
 - ii) Draw the image of the figure according the rules given.
 - a) $(x; y) \rightarrow (-x; y)$
 - b) $(x; y) \rightarrow (y; x)$
 - c) $(x; y) \rightarrow (x; y - 3)$
 - d) $(x; y) \rightarrow (x + 5; y)$
 - e) $(x; y) \rightarrow (x; -y)$
-
-

Activity :: Investigation : Calculation of Volume, Surface Area and scale factors of objects

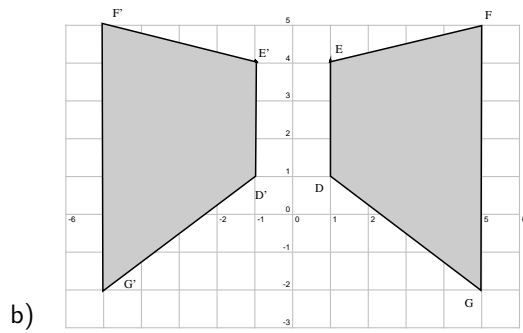
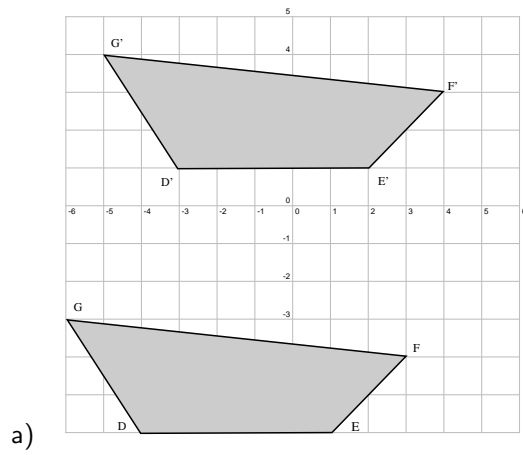
1. Look around the house or school and find a can or a tin of any kind (e.g. beans, soup, cooldrink, paint etc.)
2. Measure the height of the tin and the diameter of its top or bottom.
3. Write down the values you measured on the diagram below:



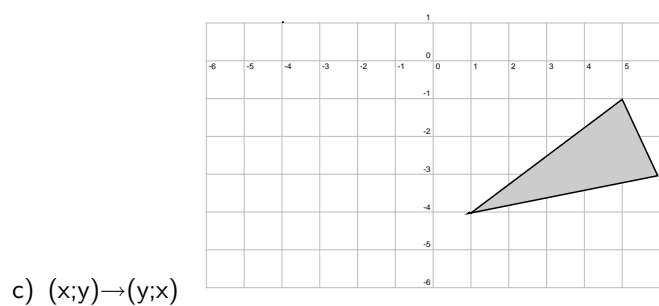
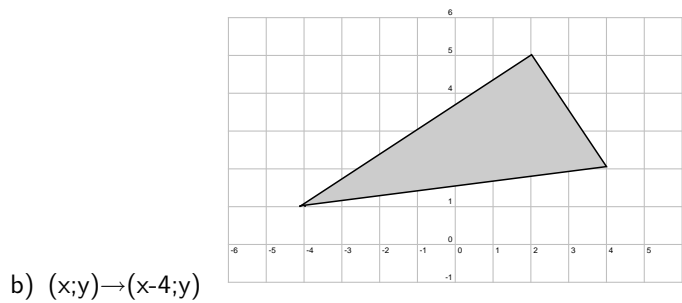
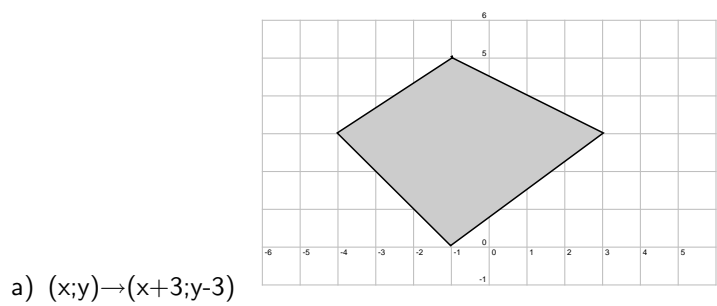
4. Using your measurements, calculate the following (in cm^2 , rounded off to 2 decimal places):
 - (a) the area of the side of the tin (i.e. the rectangle)
 - (b) the area of the top and bottom of the tin (i.e. the circles)
 - (c) the total surface area of the tin
 5. If the tin metal costs 0,17 cents/ cm^2 , how much does it cost to make the tin?
 6. Find the volume of your tin (in cm^3 , rounded off to 2 decimal places).
 7. What is the volume of the tin given on its label?
 8. Compare the volume you calculated with the value given on the label. How much air is contained in the tin when it contains the product (i.e. cooldrink, soup etc.)
 9. Why do you think space is left for air in the tin?
 10. If you wanted to double the volume of the tin, but keep the radius the same, by how much would you need to increase the height?
 11. If the height of the tin is kept the same, but now the radius is doubled, by what scale factor will the:
 - (a) area of the side surface of the tin increase?
 - (b) area of the bottom/top of the tin increase?
-

13.6 End of Chapter Exercises

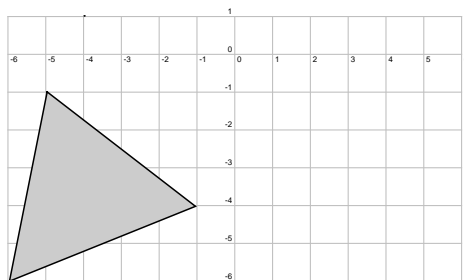
1. Write a rule that will give the following transformations of DEFG to D'E'F'G in each case.



2. Using the rules given, identify the type of transformation and draw the image of the shapes.



d) $(x;y) \rightarrow (-x;-y)$



Appendix A

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